FREE PRODUCTS, CYCLIC HOMOLOGY, AND THE GAUSS-MANIN CONNECTION

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Abstract. We present a new approach to cyclic homology that does not involve Connes' differential and is based on $(\Omega^{\bullet}A)[u], \mathsf{d} + u \cdot \imath_\Delta$, a *noncommutative equivariant de Rham complex* of an associative algebra A. Here d is the Karoubi-de Rham differential, which replaces the Connes differential, and $\imath_\Delta$ is an operation analogous to contraction with a vector field. As a byproduct, we give a simple explicit construction of the Gauss-Manin connection, introduced earlier by E. Getzler, on the relative cyclic homology of a flat family of associative algebras over a central base ring.

We introduce and study *free-product deformations* of an associative algebra, a new type of deformation over a not necessarily commutative base ring. Natural examples of free-product deformations arise from preprojective algebras and group algebras for compact surface groups.

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1. Introduction

Throughout, we fix a field $\Bbbk$ of characteristic 0 and write $\otimes = \otimes_{\Bbbk}$. By an algebra we always mean an associative unital $\Bbbk$-algebra, unless explicitly stated otherwise. Given an algebra A, we view the space $A \otimes A$ as an A-bimodule with respect to the *outer* bimodule structure, which is defined by the formula $b(a' \otimes a'')c := (ba') \otimes (a''c)$, for any $a', a'', b, c \in A$.

1.1. It is well-known that a regular vector field on a smooth affine algebraic variety X is the same thing as a derivation $\Bbbk[X] \to \Bbbk[X]$ of the coordinate ring of X. Thus, derivations of a commutative algebra A play the role of vector fields.

It has been commonly accepted until recently that this point of view applies to noncommutative algebras A as well. A first indication towards a different point of view was a discovery by Crawley-Boevey [CB] that, for a smooth affine curve X with coordinate ring $A = \Bbbk[X]$, the algebra of differential operators on X can be constructed by means of *double derivations* $A \to A\otimes A$, rather than ordinary derivations $A \to A$. Since then, the significance of double derivations in noncommutative geometry was explored further in [VdB] and [CBEG].

To explain the role of double derivations in more detail we first recall some basic definitions.

1.2. Let B be any algebra and N a B-*bimodule*. Recall that a $\Bbbk$-linear map $f : B \to N$ is said to be a derivation of B with coefficients in N if $f(b_1b_2) = f(b_1)b_2 + b_1f(b_2)$, $\forall b_1, b_2 \in B$. Given a subalgebra $R \subset B$, we let $\mathrm{Der}_R(B, N)$ denote the space of relative derivations of B with respect to the subalgebra R, that is, of derivations $B \to N$ that annihilate the subalgebra R.

Definition 1.2.1. Given $t \in B$, a $\Bbbk$-linear map $f : B \to N$ is called a *t-derivation* if

$$f(1) = 0 \quad \text{and} \quad f(b_1 \cdot t \cdot b_2) = f(b_1) \cdot t \cdot b_2 + b_1 \cdot t \cdot f(b_2), \quad \forall b_1, b_2 \in B. \tag{1.2.2}$$

We can view a t-derivation as a derivation $f : B \to N$, where B is equipped with a new *non-unital* algebra structure given by $a \circ b := a \cdot t \cdot b$, and N similarly has the modified bimodule structure $b \circ n = b \cdot t \cdot n$ and $n \circ b = n \cdot t \cdot b$. It follows, in particular, that the space of t-derivations $B \to B$ is a Lie algebra with respect to the commutator bracket. One may prove by induction on n that $f(t^n) = 0$, $\forall n = 1, 2, \ldots$. Note that a t-derivation need not be a derivation, in general.

Recall that a free product of two algebras A and B, is an associative algebra $A * B$ that contains A and B as subalgebras and whose elements are formal $\Bbbk$-linear combinations of words $a_1 b_1 a_2 b_2 \ldots a_n b_n$, for any $n \geq 1$ and $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$. These words are taken up to equivalence imposed by the relation $1_A = 1_B$; for instance, we have $\ldots b 1_A b' \ldots = \ldots b 1_B b' \ldots = \ldots (b \cdot b') \ldots$, for any $b, b' \in B$.

We are interested in the special case where $B = \Bbbk[t]$, a polynomial algebra in one variable.

Lemma 1.2.3. *For any $A * \Bbbk[t]$-bimodule M, we have*

(i) *Restriction to the subalgebra $A \subset A * \Bbbk[t]$ provides a vector space isomorphism*

$$\left\{ \begin{matrix} t\text{-derivations} \\ F\colon A*\Bbbk[t] \to M \end{matrix} \right\} \xrightarrow[\sim]{F \mapsto f := F|_A} \left\{ \begin{matrix} \Bbbk\text{-linear maps } f\colon A \to M \\ \text{such that } f(1)=0 \end{matrix} \right\}$$

(ii) *The isomorphism in* (i) *restricts to a bijection:* $\mathrm{Der}_{\Bbbk[t]}(A * \Bbbk[t], M) \xrightarrow{\sim} \mathrm{Der}_{\Bbbk}(A, M)$.

Proof. It is clear that the assignment $F \mapsto f := F|_A$ gives an injective map from the set of t-derivations $F : A * \Bbbk[t] \to M$ to the set of $\Bbbk$-linear maps $f : A \to M$ such that $f(1) = 0$. We construct a map in the opposite direction by assigning to any $\Bbbk$-linear map $f : A \to M$ such that $f(1) = 0$ a map $f_t : A * \Bbbk[t] \to M$, given, for any $a_1, \ldots, a_n \in A$, by the following Leibniz-type formula:

$$f_t : \ a_1\, t\, a_2\, t \ldots t\, a_n \longmapsto \sum\nolimits_{k=1}^{n} a_1\, t \ldots a_{k-1}\, t\, f(a_k)\, t\, a_{k+1}\, t \ldots t\, a_n. \tag{1.2.4}$$

One verifies that the map f_t thus defined satisfies (1.2.2). It is immediate to check that the maps $F \mapsto F|_A$ and $f \mapsto f_t$ are inverse to each other. This proves (i). Part (ii) is straightforward and is left to the reader. $\square$

Notation 1.2.5. We write f_t for the t-derivation (1.2.4) that corresponds to a $\Bbbk$-linear map $f : A \to M$ under the inverse to the isomorphism of Lemma 1.2.3(i).

We will use simplified notation $A_t := A * \Bbbk[t]$ and let $A_t^+ = A_t \cdot t \cdot A_t$ be the two-sided ideal of the algebra A_t generated by t. Furthermore, let $\mathrm{Der}_t(A_t) := \mathrm{Der}_{\Bbbk[t]}(A * \Bbbk[t],\ A * \Bbbk[t])$ denote the Lie algebra of derivations of the algebra $A * \Bbbk[t]$ *relative* to the subalgebra $\Bbbk[t]$.

1.3. **Derivations vs double derivations.** Recall that derivations of an algebra A may be viewed as 'infinitesimal automorphisms'. Specifically, let $A[t] = A \otimes \Bbbk[t]$ be the polynomial ring in one variable with coefficients in A. Thus, $A[t]$ is a $\Bbbk[t]$-algebra and, for any $\Bbbk$-linear map $\xi : A \to A$, the assignment $A \to A[t]$, $a \mapsto t \cdot \xi(a)$ can be uniquely extended to a $\Bbbk[t]$-linear map $t\xi : A[t] \to A[t]$.

A well-known elementary calculation yields

Lemma 1.3.1. *The following properties of a $\Bbbk$-linear map $\xi : A \to A$ are equivalent:*

- *The map ξ is a derivation of the algebra A;*
- *The map $t\xi$ is a derivation of the algebra $A[t]$ relative to the subalgebra $\Bbbk[t]$;*
- *The map* $\mathrm{Id} + t\xi : \ A[t]/t^2 \cdot A[t] \to A[t]/t^2 \cdot A[t]$ *is an algebra automorphism.*

All the above holds true, of course, no matter whether the algebra A is commutative or not. Yet, the element t, the formal parameter, is by definition a *central* element of the algebra $A[t]$.

In noncommutative geometry, the assumption that the formal parameter t be central is not quite natural, however. Thus, we are led to consider a free product algebra $A_t = A * \Bbbk[t]$, freely generated by A and an indeterminate t.

We are going to argue that, once the polynomial algebra $A[t]$ is replaced by the algebra A_t, it becomes more natural to replace derivations $A \to A$ by *double derivations*, i.e., by derivations $A \to A \otimes A$ where $A \otimes A$ is viewed as an A-bimodule with respect to the outer bimodule structure. To see this, observe that there are natural A-bimodule isomorphisms, cf. Notation 1.2.5,

$$A_t/A_t^+ \xrightarrow{\sim} A, \quad \text{and} \quad A_t/(A_t^+)^2 \xrightarrow{\sim} A \oplus (A \otimes A), \quad a + a'\, t\, a'' \longmapsto a \oplus (a' \otimes a''). \tag{1.3.2}$$

Let $\Theta : A \to A \otimes A$ be a $\Bbbk$-linear map. We will use symbolic Sweedler notation to write this map as $a \mapsto \Theta'(a) \otimes \Theta''(a)$, where we systematically suppress the summation symbol. We observe that the assignment $a \mapsto \Theta'(a)\, t\, \Theta''(a)$ gives a map $A \to A_t^+$. Assuming, in addition, that $\Theta(1) = 0$, we let $\Theta_t : A_t \to A_t$ be the associated t-derivation, see Notation 1.2.5 and Lemma 1.2.3.

Now, a free product analogue of Lemma 1.3.1 reads

Lemma 1.3.3. *The following properties of a $\Bbbk$-linear map $\Theta : A \to A \otimes A$ are equivalent:*
- *The map Θ is a double derivation;*
- *The map Θ_t is a derivation of the algebra A_t relative to the subalgebra $\Bbbk[t]$;*
- *The map $\mathrm{Id} + \Theta_t : \ A_t/(A_t^+)^2 \to A_t/(A_t^+)^2$ is an algebra automorphism.*

We see that, in noncommutative geometry, the algebra A_t should play the role of the polynomial algebra $A[t]$. Some aspects of this philosophy will be discussed further in subsequent sections.

1.4. **Layout of the paper.** In §2, we recall the definition of the DG algebra of noncommutative differential forms [Con, CC], following [CQ1], and that of the Karoubi-de Rham complex, cf. [Kar]. We also introduce an *extended Karoubi-de Rham complex*, that will play a crucial role later. In §3, we develop the basics of noncommutative calculus involving the action of double derivations on the extended Karoubi-de Rham complex, via Lie derivative and contraction operations.

In §4, we state three main results of the paper. The first two, Theorem 4.1.1 and Theorem 4.2.2, provide a description, in terms of the Karoubi-de Rham complex, of the Hochschild homology of an algebra A and of the cyclic homology of A, respectively. The third result, Theorem 4.3.1, gives a formula for the Gauss-Manin connection on periodic cyclic homology of a family of algebras, [Ge], in a way that avoids complicated formulas and resembles equivariant cohomology. These results are proved in §5, using properties of the Karoubi operator and the harmonic decomposition of noncommutative differential forms introduced by Cuntz and Quillen, [CQ1, CQ2].

In §6, we establish a connection between cyclic homology and equivariant cohomology via the representation functor. More precisely, we give a homomorphism from our noncommutative equivariant de Rham complex (which extends the complex used to compute cyclic homology) to the equivariant de Rham complex computing equivariant cohomology of the representation variety.

In §7, we introduce a new notion of free product deformation over a not necessarily commutative base. We extend classic results of Gerstenhaber concerning deformations of an associative algebra A to our new setting of free product deformations. To this end, we consider a double-graded Hochschild complex $\big(\oplus_{p,k\geq 2}\ C^p(A, A^{\otimes k}),\ \mathsf{b}\big)$. We define a new associative product $f, g \mapsto f \vee g$ on that complex, and study Maurer-Cartan equations of the form $\mathsf{b}(\beta) + \frac{1}{2}\beta \vee \beta = 0$.

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2. Extended Karoubi-de Rham complex

2.1. **The commutator quotient.** Let $B = \oplus_{k\in\mathbb{Z}} B^k$ be a $\mathbb{Z}$-graded algebra and $M = \oplus_{k\in\mathbb{Z}} M^k$ a $\mathbb{Z}$-graded B-bimodule. For a homogeneous element $u \in B^k$, we write $|u| = k$. A linear map $f : B^\bullet \to M^{\bullet+n}$ is said to be a *degree n graded* derivation if, for any homogeneous $u, v \in B$, we have $f(uv) = f(u)\cdot v + (-1)^{n|u|} u \cdot f(v)$. Let $\mathrm{Der}^n(B, M)$ denote the vector space of degree n graded derivations. The direct sum $\mathrm{Der}^\bullet B := \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}^n(B, B)$, of graded derivations of the algebra B, has a natural Lie *super*-algebra structure given by the *super*-commutator.

We write $[B, B]$ for the *super*-commutator space of a graded algebra B, the $\Bbbk$-linear span of the set $\{uv - (-1)^{pq} vu \mid u \in B^p, v \in B^q,\ p, q \in \mathbb{Z}\}$. This is a graded subspace of B, and we may consider the commutator quotient space $B_{\mathrm{cyc}} := B/[B, B]$, equipped with induced grading $B^\bullet_{\mathrm{cyc}} = \oplus_{k\in\mathbb{Z}} B^k_{\mathrm{cyc}}$. Any degree n graded derivation $f : B^\bullet \to B^{\bullet+n}$ descends to a well-defined linear map of graded vector spaces $f_{\mathrm{cyc}} : B^\bullet_{\mathrm{cyc}} \to B^{\bullet+n}_{\mathrm{cyc}}$.

We will also use the free product construction for *graded* algebras. Given a graded algebra B, the algebra $B_t = B * \Bbbk[t]$ acquires a natural grading $B^\bullet_t = \oplus_{k\in\mathbb{Z}} B^k_t$, that agrees with the one on the subalgebra $B \subset B_t$ and is such that $\deg t = 2$.

There is a graded analogue of Lemma 1.2.3. Namely, given a graded $B^\bullet_t$-bimodule $M^\bullet = \oplus_{k\in\mathbb{Z}} M^k$ and a $\Bbbk$-linear map $f : B^\bullet \to M^{\bullet+n}$ such that $f(1) = 0$, we introduce a degree n derivation $f_t : B^\bullet_t \to M^{\bullet+n}$ defined, for any homogeneous elements $u_1, \ldots, u_n \in B$, by the formula

$$f_t(u_1\, t\, u_2\, t \ldots t\, u_r) = \sum\nolimits_{k=1}^{r} (-1)^{n(|u_1|+\ldots+|u_{k-1}|)} \cdot u_1\, t \ldots u_{k-1}\, t\, f(u_k)\, t\, u_{k+1}\, t \ldots t\, u_r. \tag{2.1.1}$$

Given an algebra A and an A-bimodule N, let $T_A N = \oplus_{n\geq 0} T^n_A N$ be the tensor algebra of N over A. Thus, $T^\bullet_A N$ is a graded associative algebra with $T^0_A N = A$.

2.2. **Noncommutative differential forms.** Fix an algebra B and a subalgebra $R \subset B$. Let $\Omega^1_R B := \mathrm{Ker}(m)$ be the kernel of the multiplication map $m : B \otimes_R B \to B$, and write $i_\Delta : \Omega^1_R B \hookrightarrow B \otimes_R B$ for the tautological imbedding. Thus, $\Omega^1_R B$ is a B-bimodule, called the bimodule of *noncommutative 1-forms* on the algebra B relative to the subalgebra R, and we have a short exact sequence of B-bimodules, see [CQ1, §2],

$$0 \longrightarrow \Omega^1_R B \xrightarrow{\ i_\Delta\ } B \otimes_R B \xrightarrow{\ m\ } B \longrightarrow 0. \tag{2.2.1}$$

The assignment $b \mapsto \mathsf{d}b := 1\otimes b - b \otimes 1$ gives a canonical derivation $\mathsf{d} : B \to \Omega^1_R B$. This derivation is 'universal' in the sense that, for any B-bimodule M, we have a bijection

$$\mathrm{Der}_R(B, M) \xrightarrow{\sim} \mathrm{Hom}_{B\text{-}\mathsf{bimod}}(\Omega^1_R B, M), \quad \theta \mapsto i_\theta, \tag{2.2.2}$$

where $i_\theta : \Omega^1_R B \to M$ stands for a B-bimodule map defined by the formula $i_\theta(u\, \mathsf{d}v) := u \cdot \theta(v)$.

The tensor algebra $\Omega_R B := T^\bullet_B(\Omega^1_R B)$ of the B-bimodule $\Omega^1_R B$ is a DG algebra, $(\Omega^\bullet_R B, \mathsf{d})$, called the algebra of noncommutative differential forms on B relative to the subalgebra R (we will interchangeably use the notation $\Omega_R B$ or $\Omega^\bullet_R B$ depending on whether we want to emphasize the grading or not). For each $n \geq 1$, there is a standard isomorphism of left B-modules, see [CQ1], $\Omega^n_R B = B \otimes_R T^n_R(B/R)$; usually, one writes $b_0\, \mathsf{d}b_1\, \mathsf{d}b_2 \ldots \mathsf{d}b_n \in \Omega^n_R B$ for the n-form corresponding to an element $b_0 \otimes (b_1 \otimes \ldots \otimes b_n) \in B \otimes_R T^n_R(B/R)$ under this isomorphism. The de Rham differential $\mathsf{d} : \Omega^\bullet_R B \to \Omega^{\bullet+1}_R B$ is given by the formula $\mathsf{d} : b_0\, \mathsf{d}b_1\, \mathsf{d}b_2 \ldots \mathsf{d}b_n \mapsto \mathsf{d}b_0\, \mathsf{d}b_1\, \mathsf{d}b_2 \ldots \mathsf{d}b_n$.

Following Karoubi [Kar], we define the (relative) noncommutative de Rham complex of B as

$$\mathrm{DR}_R B := (\Omega_R B)_{\mathrm{cyc}} = \Omega_R B/[\Omega_R B, \Omega_R B],$$

the super-commutator quotient of the *graded* algebra $\Omega^\bullet_R B$. The space $\mathrm{DR}_R B$ comes equipped with a natural grading and with de Rham differential $\mathsf{d} : \mathrm{DR}^\bullet_R B \to \mathrm{DR}^{\bullet+1}_R B$, induced from the one on

$\Omega^\bullet_R B$. In degree zero, we have $\mathrm{DR}^0_R B = B/[B,B]$, where $[B,B] \subset B$ is the subspace spanned by ordinary commutators.

In the 'absolute' case $R = \Bbbk$ we will use unadorned notation $\mathrm{Der}(B,N) := \mathrm{Der}_\Bbbk(B,N)$, $\Omega^n B := \Omega^n_\Bbbk B$, $\mathrm{DR}B := \mathrm{DR}_\Bbbk B$, $TN := T_\Bbbk N$, etc.

We are going to introduce an enlargement of the noncommutative de Rham complex as follows. Fix an algebra A and put a grading on $A^{\otimes 2} \oplus \Omega^1 A$ by assigning the direct summand $\Omega^1 A$ grade degree 1 and the direct summand $A^{\otimes 2}$ grade degree 2.

Lemma 2.2.3. *There are natural graded algebra isomorphisms*

$$\Omega_{\Bbbk[t]}(A_t) \;\cong\; (\Omega A) * \Bbbk[t] \;\cong\; T_A(A^{\otimes 2} \oplus \Omega^1 A). \tag{2.2.4}$$

The differential d *on* $\Omega_{\Bbbk[t]}(A_t)$ *goes under the first isomorphism to* d_t*, the induced graded derivation on* $(\Omega A) * \Bbbk[t]$ *given by* (2.1.1).

Proof. Observe first that, for any algebra R, we have a natural DG algebra isomorphism $\Omega(A*R) \cong (\Omega A)*(\Omega R)$. Since $\Omega_R(A*R)$ is a quotient of the algebra $\Omega(A*R)$ by the two-sided ideal generated by the space $\mathsf{d}R \subset \Omega^1 R \subset \Omega^1(A*R)$, the isomorphism above induces a DG algebra isomorphism

$$\Omega_R(A*R) \cong (\Omega A) * R.$$

In the special case where $R = \Bbbk[t]$, this gives the first isomorphism of the lemma.

To prove the second isomorphism in (2.2.4), fix an A-bimodule M. View $A^{\otimes 2} \oplus M$ as an A-bimodule. The assignment $(a'\otimes a'') \oplus m \mapsto a'ta'' + m$ clearly gives an A-bimodule map $A^{\otimes 2} \oplus M \to (T_A M)_t$. This map can be extended, by the universal property of the tensor algebra, to an algebra morphism $T_A(A^{\otimes 2} \oplus M) \to (T_A M)_t$. To show that this morphism is an isomorphism, we explicitly construct an inverse map as follows.

We start with a natural algebra imbedding $f : T_A M \hookrightarrow T_A(A^{\otimes 2} \oplus M)$, induced by the A-bimodule imbedding $M = 0 \oplus M \hookrightarrow A^{\otimes 2} \oplus M$. Then, by the universal property of free products, we can (uniquely) extend the map f to an algebra homomorphism $(T_A M)_t = (T_A M) * \Bbbk[t] \to T_A(A^{\otimes 2} \oplus M)$ by sending $t \mapsto 1_A \otimes 1_A \in A^{\otimes 2} \subset T^1_A(A^{\otimes 2} \oplus M)$. It is straightforward to check that the resulting homomorphism is indeed an inverse of the homomorphism in the opposite direction constructed earlier.

Applying the above in the special case $M = \Omega^1 A$ yields the second isomorphism of the lemma. $\square$

For the DG algebra of noncommutative differential forms on the algebra $A * \Bbbk[t]$ relative to the subalgebra $\Bbbk[t]$, we introduce the special notation $\Omega_t A := \Omega_{\Bbbk[t]}(A_t)$. The *extended de Rham complex* of A is defined as a super-commutator quotient

$$\mathrm{DR}_t A := \mathrm{DR}_{\Bbbk[t]}(A_t) = (\Omega_{\Bbbk[t]}(A_t))_{\mathrm{cyc}} \cong ((\Omega A)_t)_{\mathrm{cyc}}.$$

2.3. **Gradings.** Any (non-graded) algebra may be regarded as a graded algebra concentrated in degree zero. Thus, for an algebra B without grading we have the subspace $[B,B] \subset B$ spanned by ordinary commutators, and the corresponding commutator quotient space $B_{\mathrm{cyc}} = B/[B,B]$.

In the above situation, the free product $B_t = \oplus_{k\geq 0}\, B^{2k}_t$ is an *even*-graded algebra with the grading that counts *twice* the number of occurrences of the variable t. For the corresponding commutator quotient, we get $(B^\bullet_t)_{\mathrm{cyc}} = \oplus_{k\geq 0}\, (B^k_t)_{\mathrm{cyc}}$; in particular, $(B^0_t)_{\mathrm{cyc}} = B_{\mathrm{cyc}} = B/[B,B]$. Furthermore, for any $n \geq 1$, the space $(B^{2n}_t)_{\mathrm{cyc}}$ is spanned by *cyclic* words $u_1\, t\, u_2\, t \ldots t\, u_n\, t$, where cyclic means that, for instance, $u_1\, t\, u_2\, t\, u_3\, t = t\, u_3\, t\, u_1\, t\, u_2$ (modulo commutators). In particular, we obtain an isomorphism $(B^2_t)_{\mathrm{cyc}} \cong B$.

More generally, the assignment $u_1\, t\, u_2\, t \ldots t\, u_n \mapsto u_1 \otimes u_2 \otimes \ldots \otimes u_n$ yields, for any (non-graded) algebra B, natural vector space isomorphisms

$$B^{2n}_t \xrightarrow{\sim} B^{\otimes n} \quad \text{and} \quad (B^{2n}_t)_{\mathrm{cyc}} \cong B^{\otimes(n-1)}_{\mathrm{cyc}}, \quad \forall n = 1, 2, \ldots. \tag{2.3.1}$$

Here and elsewhere, given a vector space V, we let the group $\mathbb{Z}/n\mathbb{Z}$ act on $V^{\otimes n} = T^n V$ by cyclic permutations and write

$$V^{\otimes n}_{\mathrm{cyc}} := V^{\otimes n}/(\mathbb{Z}/n\mathbb{Z}). \tag{2.3.2}$$

Next, for each integer k, let $\mathrm{Der}^{2k}_t(B_t) := \mathrm{Der}^{2k}_{\Bbbk[t]}(B_t, B_t)$ be the space of graded derivations $B^\bullet_t \to B^{\bullet+2k}_t$, relative to the subalgebra $\Bbbk[t] \subset B_t$. Thus, $\mathrm{Der}_t(B_t) = \oplus_{k\in\mathbb{Z}}\ \mathrm{Der}^{2k}_t(B_t)$ is a graded Lie super-algebra.

It is clear that, for any derivation $\theta : B \to B$, the derivation $\theta_t : B_t \to B_t$ has degree zero, i.e., we have $\theta_t \in \mathrm{Der}^0_t(B_t)$. On the other hand, for any double derivation $\Theta \in \mathbb{D}\mathrm{er}\, B$, the derivation $\Theta_t : B_t \to B_t$ has degree 2, i.e., we have $\Theta_t \in \mathrm{Der}^2_t(B_t)$. It is easy to check, using a graded version of Lemma 1.2.3, that converse statements are also true, and we have:

$$\mathrm{Der}^0_t(B_t) = \{\theta_t \mid \theta \in \mathrm{Der}\, B\}, \quad \text{and} \quad \mathrm{Der}^2_t(B_t) = \{\Theta_t \mid \Theta \in \mathbb{D}\mathrm{er}\, B\}. \tag{2.3.3}$$

Now, fix an algebra A. We observe that the algebra $\Omega_t A = \Omega_{\Bbbk[t]}(A_t)$ comes equipped with a natural *bi*-grading $\Omega_t A = \oplus_{p,q\geq 0}\, \Omega^{2p,q}_t A$, where the even p-grading is induced from the one on A_t, and the q-component corresponds to the grading induced by the natural one on $\Omega^\bullet A$. It is easy to see that the p-grading corresponds, under the isomorphism (2.2.4) to the grading on $(\Omega A) * \Bbbk[t]$ that counts twice the number of occurrences of the variable t. For example, for any $\alpha \in \Omega^k A$ and $\beta \in \Omega^\ell A$, the element $\alpha\, t\, \beta\, t \in (\Omega A) * \Bbbk[t]$ has bidegree $(2p = 4, q = k + \ell)$.

The bigrading on $\Omega_t A$ clearly descends to a bigrading on the extended de Rham complex of A. The de Rham differential has bidegree $(0, 1)$:

$$\mathrm{DR}_t A = \oplus_{p,q}\, \mathrm{DR}^{2p,q}_t A, \qquad \mathsf{d} :\ \mathrm{DR}^{2p,q}_t A \to \mathrm{DR}^{2p,q+1}_t A.$$

Furthermore, use the identification (2.3.1) for $B := \Omega A$, and equip $(\Omega^\bullet A)^{\otimes p}$ with the tensor product grading that counts the total degree of differential forms involved, e.g., given $\alpha_i \in \Omega^{k_i} A$, $i = 1, \ldots, p$, for $\alpha := \alpha_1 \otimes \ldots \otimes \alpha_p \in (\Omega A)^{\otimes p}_{\mathrm{cyc}}$, we put $\deg \alpha := k_1 + \ldots + k_p$. Then, we get

$$\mathrm{DR}^{2p,q}_t A = \begin{cases} \mathrm{DR}^q A & \text{if } p = 0; \\ \text{degree } q \text{ component of } (\Omega^\bullet A)^{\otimes p}_{\mathrm{cyc}} & \text{if } p \geq 1. \end{cases} \tag{2.3.4}$$

3. Noncommutative calculus

3.1. Fix an algebra B and a subalgebra $R \subset B$. Any derivation $\theta \in \mathrm{Der}_R B$ gives rise, naturally, to a Lie derivative map $L_\theta : \Omega^\bullet_R B \to \Omega^\bullet_R B$, and also to a contraction (with θ) map $i_\theta : \Omega^\bullet_R B \to \Omega^{\bullet-1}_R B$. The map L_θ is a degree zero derivation of the graded algebra $\Omega^\bullet_R B$. It is defined on 1-forms by the formula $L_\theta(u\,\mathsf{d}v) = (\theta(u))\,\mathsf{d}v + u\,\mathsf{d}(\theta(v))$, and then is extended uniquely to a derivation $L_\theta : \Omega^\bullet_R B \to \Omega^\bullet_R B$. The contraction map i_θ is a degree -1 graded derivation. It is defined on 1-forms by the formula following (2.2.2) and is extended to a map $\Omega^\bullet B \to \Omega^{\bullet-1} B$ as a *graded*-derivation. The maps L_θ and i_θ both descend to well-defined operations on the de Rham complex $\mathrm{DR}^\bullet_R B = (\Omega_R B^\bullet)_{\mathrm{cyc}}$.

Now, let A be an algebra and $\theta \in \mathrm{Der}\, A$ a derivation. On one hand, applying Lemma 1.2.3(ii) to the derivation $L_\theta : \Omega^\bullet A \to \Omega^\bullet A$ yields a derivation $(L_\theta)_t : (\Omega\, A)_t \to (\Omega\, A)_t$. On the other hand, we may first extend θ to a derivation $\theta_t : A_t \to A_t$, and then consider the Lie derivative L_{θ_t}, a derivation of the algebra $\Omega_{\Bbbk[t]}(A_t) = \Omega_t A$, of bidegree $(0, 0)$. Very similarly, we also have graded derivations, $(i_\theta)_t$ and i_{θ_t}, of bidegree $(0, -1)$.

It is immediate to see that the two procedures above agree with each other in the sense that, under the identification $\Omega_{\Bbbk[t]}(A_t) \cong (\Omega\, A)_t$ provided by (2.2.4), we have

$$L_{\theta_t} = (L_\theta)_t, \quad \text{and} \quad i_{\theta_t} = (i_\theta)_t. \tag{3.1.1}$$

3.2. **Lie derivative and contraction for double derivations.** Write $\mathbb{D}\mathrm{er}A := \mathrm{Der}(A, A\otimes A)$ for the vector space of double derivations $A \to A\otimes A$. Double derivations do *not* give rise to natural operations on the DG algebra $\Omega^{\bullet}A$. Instead, we have

Proposition 3.2.1. *Any double derivation $\Theta \in \mathbb{D}\mathrm{er}\, A$ gives a canonical Lie derivative operation, L_Θ, and a contraction operation, i_Θ, on the* extended *DG algebra $\Omega_t^{\bullet}A$.*

Proof. We observe that the left-hand sides of equations (3.1.1) still make sense for double derivations. In more detail, given $\Theta \in \mathbb{D}\mathrm{er}\, A$, we first extend it to a free product derivation $\Theta_t : A_t \to A_t$, as in §1.3. Hence, we have associated Lie derivative, L_{Θ_t}, and contraction, i_{Θ_t}, operations on the complex $\Omega_{\Bbbk[t]}(A_t)$, of *relative* differential forms on the algebra A_t. Thus, we may use (2.3.1) to interpret L_{Θ_t} and i_{Θ_t} as operations on $\Omega_t A$, to be denoted by L_Θ and i_Θ, respectively. □

Corollary 3.2.2. *Any double derivation $\Theta \in \mathbb{D}\mathrm{er}A$ gives rise canonically to a degree 0 double derivation $\mathbf{L}_\Theta \in \mathbb{D}\mathrm{er}^0(\Omega A)$, and a degree -1* graded *double derivation $\mathbf{i}_\Theta \in \mathbb{D}\mathrm{er}^{-1}(\Omega A)$.*

Proof. Fix $\Theta \in \mathbb{D}\mathrm{er}A$ and keep the notation of the proof of Proposition 3.2.1. Observe that the operations L_{Θ_t} and i_{Θ_t}, viewed as maps $(\Omega A)_t \to (\Omega A)_t$, are both graded derivations of degree 2 with respect to the grading that counts (twice) the number of occurrences of t (and disregards the degrees of differential forms). Hence, applying Lemma 2.3.3, we conclude that there exists a unique double derivation $\mathbf{L}_\Theta : \Omega A \to (\Omega A)\otimes(\Omega A)$ such that, for the corresponding map $(\Omega A)_t \to (\Omega A)_t$, we have $L_{\Theta_t} = (\mathbf{L}_\Theta)_t$, cf. (3.1.1).

Similarly, we may use an analogue of Lemma 2.3.3 for *graded*-derivations, and apply the above argument to the contraction operation. This way, the equation $(\mathbf{i}_\Theta)_t = i_{\Theta_t}$ uniquely defines an A-bimodule map $\mathbf{i}_\Theta : \Omega^n A \to \bigoplus_{1\le k\le n} \Omega^{k-1}A \otimes \Omega^{n-k}A,\ n = 1, 2, \ldots.$ Explicitly, we find that $\mathbf{i}_\Theta :\ \Omega A \to (\Omega A)\otimes(\Omega A)$ is an odd double derivation given, for any $\alpha_1, \ldots, \alpha_n \in \Omega^1 A$, by the formula

$$\mathbf{i}_\Theta :\ \alpha_1\alpha_2\ldots\alpha_n \longmapsto \sum_{1\le k\le n} (-1)^{k-1}\cdot\alpha_1\ldots\alpha_{k-1}\,(i'_\Theta\alpha_k)\otimes(i''_\Theta\alpha_k)\,\alpha_{k+1}\ldots\alpha_n. \tag{3.2.3}$$

In the special case $n = 1$, formula (3.2.3) gives the map $\Omega^1 A \to A\otimes A,\ \alpha \mapsto i_\Theta\alpha = (i'_\Theta\alpha)\otimes(i''_\Theta\alpha)$, that corresponds to the derivation $\Theta \in \mathbb{D}\mathrm{er}A$ via the canonical bijection (2.2.2). □

Both the Lie derivative and contraction operations on $\Omega_t A$ descend to the commutator quotient. This way, we obtain the Lie derivative L_Θ and the contraction i_Θ on $\mathrm{DR}_t A$, the extended de Rham complex. Explicitly, using isomorphisms (2.3.4), we can write the Lie derivative L_Θ and contraction i_Θ as chains of maps of the form

$$\mathrm{DR}A \longrightarrow \Omega A \longrightarrow (\Omega A)^{\otimes 2}_{\mathrm{cyc}} \longrightarrow (\Omega A)^{\otimes 3}_{\mathrm{cyc}} \longrightarrow \ldots. \tag{3.2.4}$$

There are some standard identities involving the Lie derivative and contraction operations associated with ordinary derivations $A \to A$. Similarly, for any Θ either in $\mathbb{D}\mathrm{er}A$ or in $\mathrm{Der}\, A$, the Lie derivative and contraction operations on $\Omega_t A$ resulting from Proposition 3.2.1 satisfy:

$$L_\Theta = \mathsf{d}\circ i_\Theta + i_\Theta\circ\mathsf{d},\quad i_\Theta\circ i_\Phi + i_\Phi\circ i_\Theta = 0,\quad i_\xi\circ i_\Theta + i_\Theta\circ i_\xi = 0,\qquad \forall\Phi \in \mathbb{D}\mathrm{er}A,\ \xi \in \mathrm{Der}\, A. \tag{3.2.5}$$

It follows, in particular, that the Lie derivative L_Θ commutes with the de Rham differential d.

To prove (3.2.5), one first verifies these identities on the generators of the algebra $\Omega_t A = (\Omega A)_t$, that is, on differential forms of degrees 0 and 1, which is a simple computation. The general case then follows by observing that any commutation relation between (graded)-derivations that holds on generators of the algebra holds true for all elements of the algebra.

It is immediate that the induced operations on $\mathrm{DR}_t A$ also satisfy (3.2.5).

3.3. **Reduced Lie derivative and contraction.** Recall that the extended de Rham complex $\mathrm{DR}_t A$ has a natural bigrading (§2.3). The second component (the q-component) of that bigrading induces a grading on each of the spaces $(\Omega A)^{\otimes k}_{\mathrm{cyc}}$, $k = 1, 2, \ldots$, appearing in (3.2.4).

Fix $\Theta \in \mathbb{D}\mathrm{er}\, A$. It is clear from definitions that the maps L_Θ and i_Θ in (3.2.1) are graded derivations of bidegrees $(2, 0)$ and $(2, -1)$, respectively. We conclude that, in the Lie derivative case, all maps in the corresponding chain (3.2.4) preserve the q-grading while, in the contraction case, all maps in the corresponding chain (3.2.4) decrease the q-grading by one.

The leftmost map in (3.2.4), to be denoted $\imath_\Theta$ in the contraction case and $\mathscr{L}_\Theta$ in the Lie derivative case, will be especially important for us. These maps, to be called *reduced contraction* and *reduced Lie derivative*, respectively, have the form

$$\imath_\Theta : \mathrm{DR}^\bullet A \longrightarrow \Omega^{\bullet-1} A, \quad \text{and} \quad \mathscr{L}_\Theta : \mathrm{DR}^\bullet A \longrightarrow \Omega^\bullet A. \tag{3.3.1}$$

Explicitly, we see from (3.2.3) that the operation $\imath_\Theta$, for instance, is given, for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega^1 A$, by the following formula:

$$\imath_\Theta(\alpha_1\alpha_2\ldots\alpha_n) = \sum_{k=1}^{n} (-1)^{(k-1)(n-k+1)} \cdot (i''_\Theta \alpha_k) \cdot \alpha_{k+1} \ldots \alpha_n\, \alpha_1 \ldots \alpha_{k-1} \cdot (i'_\Theta \alpha_k). \tag{3.3.2}$$

An *ad hoc* definition of the maps in (3.3.1) via explicit formulas like (3.3.2) was first given in [CBEG]. Proving the commutation relations (3.2.5) using explicit formulas is, however, very painful; this was carried out in [CBEG] by rather long brute-force computations. Our present approach based on the free product construction yields the commutation relations for free.

3.4. **The derivation Δ.** There is a distinguished double derivation

$$\Delta : A \to A \otimes A, \quad a \mapsto 1 \otimes a - a \otimes 1.$$

The corresponding contraction map $i_\Delta : \Omega^1 A \to A \otimes A$ is the tautological imbedding (2.2.1). Furthermore, the derivation $\Delta_t : A_t \to A_t$ associated with Δ by formula (1.2.4) equals $\mathrm{ad}\, t : u \mapsto t \cdot u - u \cdot t$. Hence, the Lie derivative map $\mathbf{L}_\Delta : \Omega_t A \to \Omega_t A$ reads $\omega \mapsto \mathrm{ad}\, t(\omega) := t \cdot \omega - \omega \cdot t$.

Lemma 3.4.1. (i) *For any $a_0, a_1, \ldots, a_n \in A$, we have*

$$\imath_\Delta(a_0\, \mathsf{d}a_1 \ldots \mathsf{d}a_n) = \sum_{1 \le k \le n} (-1)^k \cdot \mathrm{ad}\, a_k(\mathsf{d}a_{k+1} \ldots \mathsf{d}a_n\, a_0\, \mathsf{d}a_1 \ldots \mathsf{d}a_{k-1}).$$

(ii) *In $\Omega^\bullet A$, we have $\imath_\Delta \circ \mathsf{d} + \mathsf{d} \circ \imath_\Delta = 0$ and $\mathsf{d}^2 = (\imath_\Delta)^2 = 0$; similar equations also hold in $\mathrm{DR}^\bullet_t A$, with i_Δ in place of $\imath_\Delta$.*

Proof. Part (i) is verified by a straightforward computation based on formula (3.2.3). We claim next that, in $\Omega_t A$, we have $L_\Delta = \mathrm{ad}\, t$. Indeed, it suffices to check this equality on the generators of the algebra Ω_t. It is clear that $L_\Delta(t) = 0 = \mathrm{ad}\, t(t)$, and it is easy to see that both derivations agree on 0-forms and on 1-forms. This proves the claim. Part (ii) of the lemma now follows by the Cartan formula on the left of (3.2.5), since the equation $L_\Delta = \mathrm{ad}\, t$ clearly implies that the map $L_\Delta : \mathrm{DR}_t A \to \mathrm{DR}_t A$, as well as the map $\mathscr{L}_\Delta$, vanish. $\square$

For any algebra A, let $A_\tau := A * \Bbbk[\tau]$ be a graded algebra such that A is placed in degree zero and τ is an *odd* variable placed in degree 1. Let $\frac{d}{d\tau}$ be a degree -1 derivation of the algebra A_τ that annihilates A and is such that $\frac{d}{d\tau}(\tau) = 1$. Similarly, let $\tau^2 \frac{d}{d\tau}$ be a degree $+1$ graded derivation of the algebra A_τ that annihilates A and is such $\tau^2 \frac{d}{d\tau}(\tau) = \tau^2$. For any homogeneous element $x \in A_\tau$, put $\mathrm{ad}\, \tau(x) := \tau x - (-1)^{|x|} \tau$; in particular, one finds that $\mathrm{ad}\, \tau(\tau) = 2\tau^2$.

It is easy to check that each of the derivations $\frac{d}{d\tau}$, $\tau^2 \frac{d}{d\tau}$, and $\mathrm{ad}\, \tau - \tau^2 \frac{d}{d\tau}$ squares to zero.

Claim 3.4.2. (i) *The following assignment gives a graded algebra imbedding:*

$$j:\ \Omega_t A \ \hookrightarrow\ A*\Bbbk[\tau],\quad t\mapsto\tau^2,\quad a_0\,\mathsf{d}a_1\ldots\mathsf{d}a_n\mapsto a_0\cdot[\tau,a_1]\cdot\ldots\cdot[\tau,a_n],$$

Moreover, the above map intertwines the contraction operation $\mathbf{i}_\Delta$ *with the differential* $\tau^2\frac{d}{d\tau}$*, and the Karoubi-de Rham differential* d *with the differential* $\operatorname{ad}\tau-\tau^2\frac{d}{d\tau}$*.*

(ii) *The image of the map* j *is annihilated by the derivation* $\frac{d}{d\tau}$*.*

(iii) *The complex* $\big((A_\tau)_{cyc},\ \frac{d}{d\tau}\big)$ *computes cyclic homology of the algebra* A. □

We will neither use nor prove this result; cf. [CQ1, Proposition 1.4] and [KS, §4.1 and Lemma 4.2.1].

4. Applications to Hochschild and cyclic homology

4.1. **Hochschild homology.** Given an algebra A and an A-bimodule M, we let $H_k(A,M)$ denote the k-th Hochschild homology group of A with coefficients in M. Also, write $[A,M]\subset M$ for the $\Bbbk$-linear span of the set $\{am-ma\mid a\in A, m\in M\}$. Thus, we have $H_0(A,M)=M/[A,M]$.

We extend some ideas of Cuntz and Quillen [CQ2] to obtain our first important result.

Theorem 4.1.1. *For any unital* $\Bbbk$*-algebra* A*, there is a natural graded space isomorphism*

$$H_\bullet(A,A)\cong\operatorname{Ker}[\imath_\Delta:\mathrm{DR}^\bullet A\to\Omega^{\bullet-1}A].$$

To put Theorem 4.1.1 in context, recall that Cuntz and Quillen used noncommutative differential forms to compute Hochschild homology. Specifically, following [CQ1] and [CQ2], consider a complex $\ldots\xrightarrow{\mathsf{b}}\Omega^2A\xrightarrow{\mathsf{b}}\Omega^1A\xrightarrow{\mathsf{b}}\Omega^0A\longrightarrow 0$. Here, b is the *Hochschild differential* given by the formula

$$\mathsf{b}:\ \alpha\,\mathsf{d}a\longmapsto(-1)^n\cdot[\alpha,a],\qquad\forall a\in A/\Bbbk,\ \alpha\in\Omega^nA,\ n>0. \tag{4.1.2}$$

It was shown in [CQ2] that the complex $(\Omega^\bullet A,\mathsf{b})$ can be identified with the standard Hochschild chain complex. It follows that $H_\bullet(\Omega A,\mathsf{b})=H_\bullet(A,A)$ are the Hochschild homology groups of A.

As will be explained later (see discussion after Proposition 4.4.1), Theorem 4.1.1 is an easy consequence of Proposition 4.4.1; the latter proposition will be proved in §5.1 below.

Remark 4.1.3. A somewhat more geometric interpretation of Theorem 4.1.1, from the point of view of representation functors, is provided by the map (6.2.6): see Theorem 6.2.5 of §6 below.

4.2. **Cyclic homology.** We recall some standard definitions, following [Lo, ch. 2 and p.162]. For any graded vector space $M=\oplus_{i\geq 0}M^i$, we introduce a $\mathbb{Z}$-graded $\Bbbk[t,t^{-1}]$-module

$$M\hat{\otimes}\Bbbk[t,t^{-1}]\ :=\ \bigoplus_{n\in\mathbb{Z}}\ \Big(\prod\nolimits_{i\in\mathbb{Z}}\ t^i\,M^{n-2i}\Big), \tag{4.2.1}$$

where the grading is such that the space $M\subset M\hat{\otimes}\Bbbk[t,t^{-1}]$ has the natural grading, and $|t|:=2$.

Below, we will use a complex of *reduced* differential forms, defined by setting $\overline{\Omega}^0:=\Omega^0A/\Bbbk=A/\Bbbk$ and $\overline{\Omega}^k:=\Omega^kA$, for all $k>0$. Let $\overline{\Omega}^\bullet:=\bigoplus_{k\geq 0}\overline{\Omega}^k$. The Hochschild differential induces a $\Bbbk[t,t^{-1}]$-linear differential $\mathsf{b}:\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]\to\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$ of degree -1.

We also have Connes' differential $\mathsf{B}:\overline{\Omega}^\bullet\to\overline{\Omega}^{\bullet+1}$, see [Con]. Following Loday and Quillen [LQ], we extend it to a $\Bbbk[t,t^{-1}]$-linear differential on $\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$ of degree $+1$. It is known that $\mathsf{B}^2=\mathsf{b}^2=0$ and $\mathsf{B}\circ\mathsf{b}+\mathsf{b}\circ\mathsf{B}=0$. Thus, the map $\mathsf{B}+t\cdot\mathsf{b}:\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]\to\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$ gives a degree $+1$ differential on $\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$.

Write $HP_{-\bullet}(-)$, where, '$-\bullet$' denotes *inverting* the degrees, for the *reduced periodic cyclic homology* as defined in [LQ] or [Lo, §5.1], using a complex with differential of degree -1. According to [CQ2], the groups $HP_{-\bullet}(A)$ turn out to be isomorphic to homology groups of the complex $(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}],\mathsf{B}+t\cdot\mathsf{b})$, with differential of degree $+1$ (which is why we must invert the degrees). It is known that the action of multiplication by t yields periodicity isomorphisms $HP_\bullet(A)\cong HP_{\bullet+2}(A)$.

Thus, we have, up to isomorphism, only the two groups $HP_{\text{even}}(A) := HP_0(A)$, and $HP_{\text{odd}}(A) := HP_1(A)$.

Furthermore, we recall the map $\imath_\Delta : \mathrm{DR}^\bullet A \to \Omega^{\bullet-1}A$. We compose it with the natural projection $\Omega^\bullet A \twoheadrightarrow \mathrm{DR}^\bullet A$ to obtain a map $\Omega^\bullet A \to \Omega^{\bullet-1}A$. The latter map descends to a map $\overline{\Omega}^\bullet \to \overline{\Omega}^{\bullet-1}$. Furthermore, we may extend this last map, as well as the de Rham differential $\mathsf{d} : \overline{\Omega}^\bullet \to \overline{\Omega}^{\bullet+1}$, to $\Bbbk[t,t^{-1}]$-linear maps $\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}] \to \overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$, of degrees -1 and $+1$, respectively.

The resulting maps d and $\imath_\Delta$ satisfy $\mathsf{d}^2 = (\imath_\Delta)^2 = 0$ and $\mathsf{d}\circ\imath_\Delta + \imath_\Delta\circ d = 0$, by Lemma 3.4.1(ii). Thus, the map $\mathsf{d} + t\cdot\imath_\Delta$ gives a degree $+1$ differential on $\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$. This differential may be thought of as some sort of equivariant differential for the 'vector field' Δ.

The following theorem, to be proved in §5.3 below, is one of the main results of the paper. It shows the importance of the reduced contraction map $\imath_\Delta$ for cyclic homology.

Theorem 4.2.2. *The homology of the complex* $(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}], \mathsf{d} + t\cdot\imath_\Delta)$ *is isomorphic to* $HP_{-\bullet}(A)$*, the reduced periodic cyclic homology of* A *(with inverted degrees).*

Remark 4.2.3 (Hodge filtration)*.* In [Ko, §1.17], Kontsevich considers a 'Hodge filtration' on periodic cyclic homology. In terms of Theorem 4.2.2, the Hodge filtration $\mathsf{F}^\bullet_{\text{Hodge}}$ may be defined as follows:

- $\mathsf{F}^n_{\text{Hodge}}HP_{\text{even}}$ consists of those classes representable by sums $\sum_{i\geq n} t^{-i}\gamma_{2i}, \quad \gamma_{2i}\in\overline{\Omega}^{2i}$;
- $\mathsf{F}^{n+\frac{1}{2}}_{\text{Hodge}}HP_{\text{odd}}$ consists of those classes representable by sums $\sum_{i\geq n} t^{-i}\gamma_{2i+1}, \quad \gamma_{2i+1}\in\overline{\Omega}^{2i+1}$.

4.3. **Gauss-Manin connection.** It is well-known that, given a smooth family $p : \mathscr{X} \to S$ of complex proper schemes over a smooth base S, there is a canonical flat connection on the relative algebraic de Rham cohomology groups $H^\bullet_{DR}(\mathscr{X}/S)$, called the *Gauss-Manin connection.* More algebraically, let A be a commutative flat $\Bbbk$-algebra over a regular subalgebra $B \subset A$. In such a case, the relative algebraic de Rham cohomology may be identified with $HP^B_\bullet(A)$, the relative periodic cyclic homology; see, e.g., [FT]. The Gauss-Manin connection therefore provides a flat connection on the relative periodic cyclic homology.

In [Ge], Getzler extended the definition of Gauss-Manin connection to a noncommutative setting. Specifically, let A be a (not necessarily commutative) associative algebra equipped with a *central* algebra imbedding $B = \Bbbk[x_1,\ldots,x_n] \hookrightarrow A$. Assuming that A is free as a B-module, Getzler has defined a flat connection on $HP^B_\bullet(A)$. Unfortunately, Getzler's definition of the connection involves quite complicated calculations in the Hochschild complex that make it difficult to relate his definition with the classical geometric construction of the Gauss-Manin connection on de Rham cohomology. Alternative approaches to the definition of Getzler's connection, also based on homological algebra, were suggested more recently by Kaledin [Kal] and by Tsygan [T].

Below, we propose a new, geometrically transparent approach for the Gauss-Manin connection using the construction of cyclic homology from the previous subsection. Unlike earlier constructions, our formula for the connection on periodic cyclic homology is identical, essentially, to the classic formula for the Gauss-Manin connection in de Rham cohomology, though the objects involved have different meanings.

Our version of Getzler's result reads as follows:

Theorem 4.3.1. *Let* B *be a commutative algebra. Let* A *be an associative algebra equipped with a central algebra imbedding* $B \hookrightarrow A$ *such that the quotient* A/B *is a free* B*-module.*

Then, there is a canonical flat connection ∇_{GM} *on* $HP^B_\bullet(A)$*.*

Notation 4.3.2. (i) Given an algebra R and a subset $J \subset R$, let (J) denote the two-sided ideal in R generated by the set J.

(ii) For a commutative algebra B, we set $\Omega^\bullet_{\text{comm}}B := \wedge^\bullet_B(\Omega^1_{\text{comm}}B)$, the super-commutative DG algebra of differential forms, generated by the B-module $\Omega^1_{\text{comm}}B$ of Kähler differentials.

Construction of the Gauss-Manin connection. Given a *central* algebra imbedding $B \hookrightarrow A$, we may realize the periodic cyclic homology of A over B as follows. First, we define the following quotient DG algebras of $(\Omega^\bullet A, \mathsf{d})$:

$$\Omega^B A := \Omega^\bullet A/([\Omega^\bullet A, \Omega^\bullet B]), \qquad \Omega(A;B) := \Omega^B A/(\mathsf{d}B).$$

Thus, we have a *central* DG algebra imbedding $\Omega^\bullet_{\mathrm{comm}} B \hookrightarrow \Omega^B A$ induced by the natural imbedding $\Omega^\bullet B \hookrightarrow \Omega^\bullet A$. We introduce the descending filtration $F^\bullet(\Omega^B A)$ by powers of the ideal $(\mathsf{d}B)$. For the corresponding associated graded algebra, we have a natural surjection

$$\Omega^\bullet(A;B)\otimes_B \Omega^i_{\mathrm{comm}} B \twoheadrightarrow \mathrm{gr}^i_F\, \Omega^B A, \quad \alpha\otimes\beta \mapsto \alpha\beta, \quad \forall \alpha \in \Omega^B A,\ \beta \in \Omega^i_{\mathrm{comm}} B. \tag{4.3.3}$$

Below, we will also make use of the objects $\overline{\Omega}^B A$ and $\overline{\Omega}(A;B)$, obtained by killing $\Bbbk \subset A = \Omega^0 A$. Thus, $\overline{\Omega}(A;B)\hat{\otimes}\Bbbk[t,t^{-1}]$ and $\overline{\Omega}^B A\hat{\otimes}\Bbbk[t,t^{-1}]$ are modules over $k[t,t^{-1}]$. There is a natural descending filtration $F^\bullet$ on $\overline{\Omega}^B A\hat{\otimes}\Bbbk[t,t^{-1}]$ induced by $F^\bullet(\Omega^B A)$ and such that $\Bbbk[t,t^{-1}]$ is placed in filtration degree zero. This filtration is obviously stable under the differential d. It is also stable under the differential $t\imath_\Delta$ since the commutators that appear in $t\imath_\Delta(\omega)$ (see Lemma 3.4.1(i)) vanish, by definition of $\Omega^B A$. Therefore, the map (4.3.3) induces a morphism of double complexes, equipped with the differentials $d\otimes_B \mathrm{Id}$ and $t\imath_\Delta\otimes_B\mathrm{Id}$,

$$\overline{\Omega}^\bullet(A;B)\hat{\otimes}\Bbbk[t,t^{-1}] \otimes_B \Omega^i_{\mathrm{comm}} B \ \twoheadrightarrow\ \mathrm{gr}^i_F\, \overline{\Omega}^B A\hat{\otimes}\Bbbk[t,t^{-1}]. \tag{4.3.4}$$

We will show in §5.5 below that the assumptions of Theorem 4.3.1 ensure that the map (4.3.3) is an isomorphism.

Assume this for the moment and consider the standard spectral sequence associated with the filtration $F^\bullet(\overline{\Omega}^B A\hat{\otimes}\Bbbk[t,t^{-1}])$. The first page of this sequence consists of terms $\mathrm{gr}_F(\overline{\Omega}^B A\hat{\otimes}\Bbbk[t,t^{-1}])$. Under the above assumption, the LHS of (4.3.4), summed over all i, composes the first page of the spectral sequence of $\Big(F^\bullet(\overline{\Omega}^B A\hat{\otimes}\Bbbk[t,t^{-1}]),\ \mathsf{d}+t\imath_\Delta\Big)$. Then, for the second page of the spectral sequence we get

$$E^2 = H^\bullet\big(\overline{\Omega}(A;B)\hat{\otimes}\Bbbk[t,t^{-1}],\ \mathsf{d}+t\imath_\Delta\big) \otimes_B \Omega^\bullet_{\mathrm{comm}} B.$$

We now describe the differential ∇ on the second page. Let

$$\nabla_{GM}:\ H^\bullet(\overline{\Omega}(A;B)\hat{\otimes}\Bbbk[t,t^{-1}]) \ \longrightarrow\ H^\bullet\big(\overline{\Omega}(A;B)\hat{\otimes}\Bbbk[t,t^{-1}]\big) \otimes_B \Omega^1_{\mathrm{comm}} B \tag{4.3.5}$$

be the restriction of ∇ to degree zero. Then we immediately see that

$$\nabla(\alpha\otimes\beta) = \nabla_{GM}(\alpha)\wedge\beta + \alpha\otimes(d_{\mathrm{DR}}\beta), \qquad (b\alpha) = b\nabla_{GM}(\alpha) + \alpha\otimes(d_{\mathrm{DR}} b), \qquad \forall b \in B,$$

where now d_{DR} is the usual de Rham differential. From these equations, we deduce that the map ∇_{GM}, from (4.3.5), gives a flat connection on $H^i\big(\overline{\Omega}(A;B)\hat{\otimes}\Bbbk[t,t^{-1}]\big)$ for all i.

Explicitly, we may describe the connection ∇_{GM} as follows. Suppose that $\bar\alpha \in \overline{\Omega}(A;B)$ has the property that $(\mathsf{d}+t\imath_\Delta)(\bar\alpha) = 0$. Let $\alpha \in \overline{\Omega}^B A$ be any lift, and consider $(\mathsf{d}+t\imath_\Delta)(\alpha)$. This must lie in $(\mathsf{d}B)$, and its image in $\Omega(A;B)\otimes_B\Omega^1_{\mathrm{comm}} B$ is the desired element. $\Diamond$

Remark 4.3.6. In [Ge], Getzler takes $B = \Bbbk[\![x_1,\ldots,x_n]\!]$, and takes A to be a *formal* deformation over B of an associative algebra A_0. Although such a setting is not formally covered by Theorem 4.3.1, our construction of the Gauss-Manin connection still applies.

To explain this, write $\mathfrak{m} \subset B = \Bbbk[\![x_1,\ldots,x_n]\!]$ for the augmentation ideal of the formal power series without constant term. Let A_0 be a $\Bbbk$-vector space with a fixed nonzero element 1_A, and let $A = A_0[\![x_1,\ldots,x_n]\!]$ be the B-module of formal power series with coefficients in A_0. We equip B and A with the $\mathfrak{m}$-adic topology, and view B as a B-submodule in A via the imbedding $b \mapsto b\cdot 1_A$.

Corollary 4.3.7. *Let $\star : A \times A \to A$ be a B-bilinear, continuous associative (not necessarily commutative) product that makes 1_A the unit element. Then, the conclusion of Theorem 4.3.1 holds for $HP^B_\bullet(A)$.*

4.4. **The Karoubi operator.** For any algebra A and an A-bimodule M, we put $M_\natural := M/[A, M] = H_0(A, M)$. Now, let $A \to B$ be an algebra homomorphism. Then, B may be viewed as an A-bimodule, and we have a canonical projection $B_\natural = B/[A, B] \twoheadrightarrow B_{\mathrm{cyc}} = B/[B, B]$. In particular, for $B = \Omega^\bullet A$, we get a natural projection $(\Omega^\bullet A)_\natural \to \mathrm{DR}^\bullet A$, which is not an isomorphism, in general.

Following Cuntz and Quillen [CQ2], we consider a diagram

$$\Omega^0 A \underset{\mathsf{b}}{\overset{\mathsf{d}}{\rightleftarrows}} \Omega^1 A \underset{\mathsf{b}}{\overset{\mathsf{d}}{\rightleftarrows}} \Omega^2 A \underset{\mathsf{b}}{\overset{\mathsf{d}}{\rightleftarrows}} \cdots.$$

Here, the de Rham differential d and the Hochschild differential b, defined in (4.1.2), are related via an important *Karoubi operator* $\kappa : \Omega^\bullet A \to \Omega^\bullet A$ [Kar]. The latter is defined by the formula $\kappa : \alpha\, \mathsf{d}a \mapsto (-1)^{\deg \alpha}\, \mathsf{d}a\, \alpha$ if $\deg \alpha > 0$, and $\kappa(\alpha) = \alpha$ if $\alpha \in \Omega^0 A$. By [Kar],[CQ1],

$$\mathsf{b} \circ \mathsf{d} + \mathsf{d} \circ \mathsf{b} = \mathrm{Id} - \kappa.$$

It follows that κ commutes with both d and b. Furthermore, it is easy to verify, cf. [CQ1] and proof of Lemma 4.4.2 below, that the Karoubi operator descends to a well-defined map $\kappa : (\Omega^n A)_\natural \to (\Omega^n A)_\natural$, which is essentially a cyclic permutation; specifically, in $(\Omega^n A)_\natural$, we have

$$\kappa(\,\alpha_1\, \alpha_2\, \ldots\, \alpha_{n-1}\, \alpha_n\,) = (-1)^{n-1} \alpha_n\, \alpha_1\, \alpha_2\, \ldots\, \alpha_{n-1}, \quad \forall \alpha_1, \ldots, \alpha_n \in \Omega^1 A.$$

Let $(-)^\kappa$ denote taking κ-invariants. In particular, write $(\Omega^\bullet A)^\kappa_\natural := [(\Omega^\bullet A)_\natural]^\kappa \subset (\Omega^\bullet A)_\natural$.

Proposition 4.4.1. *For any $n \geq 1$, we have an equality*

$$\imath_\Delta = (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ \mathsf{b} \quad \text{as maps} \quad \Omega^n A \to \Omega^{n-1} A.$$

Furthermore, the map $\imath_\Delta$ fits into a canonical short exact sequence

$$0 \longrightarrow H^n(\Omega A, \mathsf{b}) \longrightarrow \mathrm{DR}^n A \xrightarrow{\imath_\Delta} [A, \Omega^{n-1} A]^\kappa \longrightarrow 0.$$

We recall that the cohomology group $H^n(\Omega A, \mathsf{b})$ that occurs in the above displayed short exact sequence is isomorphic, as has been mentioned in §4.1, to the Hochschild homology $H_n(A, A)$. Thus, Theorem 4.1.1 is an immediate consequence of the short exact sequence of the Proposition.

The following result, which was implicit in [CQ2] and [Lo, §2.6], will play an important role in §5 below.

Lemma 4.4.2. (i) *The projection $(\Omega^\bullet A)_\natural \to \mathrm{DR}^\bullet A$ restricts to a **bijection** $(\Omega^\bullet A)^\kappa_\natural \xrightarrow{\sim} \mathrm{DR}^\bullet A$.*

(ii) *The map b descends to a map $\mathsf{b}_\natural : (\Omega^\bullet A)_\natural \to \Omega^{\bullet-1} A$.*

(iii) *The kernel of the map $\mathsf{b}_\natural : (\Omega^\bullet A)^\kappa_\natural \to \Omega^{\bullet-1} A$, the restriction of $\mathsf{b}_\natural$ to the space of κ-invariants, is isomorphic to $H^n(\Omega A, \mathsf{b})$.*

Both Proposition 4.4.1 and Lemma 4.4.2 will be proved later, in §5.1.

4.5. **Special case: $H_1(A, A)$.** For 1-forms, the formula of Proposition 4.4.1 gives $\imath_\Delta = \mathsf{b}$. Thus, using the identification $H_1(A, A) = H^1(\Omega^\bullet A, \mathsf{b})$, the short exact sequence of Proposition 4.4.1 reads

$$0 \longrightarrow H_1(A, A) \longrightarrow \mathrm{DR}^1 A \xrightarrow{\mathsf{b} = \imath_\Delta} [A, A] \longrightarrow 0. \tag{4.5.1}$$

The short exact sequence (4.5.1) may be obtained in an alternate way as follows. We apply the right exact functor $(-)_\natural$ to (2.2.1). The corresponding long exact sequence of Tor-groups reads

$$\ldots \to H_1(A, A \otimes A) \to H_1(A, A) \to (\Omega^1 A)_\natural \to (A \otimes A)_\natural \xrightarrow{c} A_\natural \to 0.$$

Now, by the definition of Tor, $H_k(A, A\otimes A) = 0$ for all $k > 0$. Also, we have natural identifications $(\Omega^1 A)_\natural = \mathrm{DR}^1 A$ and $(A\otimes A)_\natural \cong A$. This way, the map c on the right of the displayed formula above may be identified with the natural projection $A \twoheadrightarrow A/[A,A]$. Thus, $\mathrm{Ker}(c) = [A,A]$, and the long exact sequence above reduces to the short exact sequence (4.5.1).

It is immediate from definitions that the map $\mathsf{b} = \imath_\Delta$ in (4.5.1) is given by Quillen's formula $u\,\mathsf{d}v \mapsto [u,v]$ [CQ1]. In particular, we deduce that $(\mathrm{DR}^1 A)_{\mathrm{exact}} \subset \mathrm{Ker}(\imath_\Delta) = H_1(A,A)$.

4.6. **An application.** An algebra A is said to be *connected* if the following sequence is exact:

$$0 \longrightarrow \Bbbk \longrightarrow \mathrm{DR}^0 A \xrightarrow{\mathsf{d}} \mathrm{DR}^1 A. \tag{4.6.1}$$

Proposition 4.6.2. *Let A be a* connected *algebra such that $H_2(A,A) = 0$. Then,*

- $H_1(A,A) = (\mathrm{DR}^1 A)_{\mathrm{closed}} = (\mathrm{DR}^1 A)_{\mathrm{exact}}$.
- *There is a natural vector space isomorphism* $(\mathrm{DR}^2 A)_{\mathrm{closed}} \xrightarrow{\sim} [A,A]$.

Proof. We will freely use the notation of [CBEG, §4.1]. According to [CBEG, Proposition 4.1.4], for any connected algebra A, we have a map $\widetilde{\mu_{\mathrm{nc}}}$, a lift of the *noncommutative moment map*, that fits into the following commutative diagram:

$$\begin{tikzcd} \mathrm{DR}^1 A \arrow[r, "\mathsf{d}"] \arrow[d, "\imath_\Delta"'] & (\mathrm{DR}^2 A)_{\mathrm{closed}} \arrow[d, "\imath_\Delta"] \arrow[dl, dotted, "\widetilde{\mu_{\mathrm{nc}}}"'] \\ {[A,A]} \arrow[r, hook, "\mathsf{d}"] & {[A,\Omega^1 A].} \end{tikzcd} \tag{4.6.3}$$

Assuming that $H_2(A,A) = 0$, we deduce from the short exact sequence of Proposition 4.4.1 for $n = 2$ that the map $\imath_\Delta : \mathrm{DR}^2 A \to [A,\Omega^1 A]$ is injective.

We now exploit diagram (4.6.3). Since A is connected, the map d in the bottom row of the diagram is injective, by (4.6.1). Furthermore, the left vertical map $\imath_\Delta$ in the diagram is surjective by (4.5.1). Therefore, using the commutativity of (4.6.3), we deduce by diagram chase that the upper horizontal map d must be surjective, and that the map $\widetilde{\mu_{\mathrm{nc}}}$ must be bijective. This yields both statements of Proposition 4.6.2. □

A version of Proposition 4.6.2 applies in the case where A is the path algebra of a quiver with r vertices. In that case, we need to consider algebras over a ground ring $R := \Bbbk\oplus\ldots\oplus\Bbbk$ (r copies) rather than over the base field $\Bbbk$. The corresponding formalism has been worked out in [CBEG].

An analogue of Proposition 4.6.2 implies the following result, where $[A,\Omega^1_R A]^R$ stands for the the vector space formed by the elements of $[A,\Omega^1_R A]$ which commute with R.

Corollary 4.6.4. *Let A be the path algebra of a quiver. Then, there is a natural vector space isomorphism* $(\mathrm{DR}^2_R A)_{\mathrm{closed}} \xrightarrow{\sim} [A,A]^R$. □

5. Proofs

5.1. **Proof of Lemma 4.4.2 and Proposition 4.4.1.** Our proof of Lemma 4.4.2 follows the proof of [Lo, Lemma 2.6.8]. Write $\Omega^n := \Omega^n A$ and $\Omega := \oplus_n \Omega^n$.

From definitions, we get $[A,\Omega] = \mathsf{b}\Omega$ and $[\mathsf{d}A,\Omega] = (\mathrm{Id}-\kappa)\Omega$. Hence, we obtain, cf. [CQ1]:

$$[\Omega,\Omega] = [A,\Omega] + [\mathsf{d}A,\Omega] = \mathsf{b}\Omega + (\mathrm{Id}-\kappa)\Omega.$$

We deduce that $\Omega_\natural = \Omega/\mathsf{b}\Omega$, and $\mathrm{DR}^\bullet A = \Omega/[\Omega,\Omega] = \Omega_\natural/(\mathrm{Id}-\kappa)\Omega_\natural$. In particular, since $\mathsf{b}^2 = 0$, the map b descends to a well defined map $\mathsf{b}_\natural : \Omega_\natural = \Omega/\mathsf{b}\Omega \to \Omega$.

Further, one has the following standard identities [CQ2, §2]:

$$\kappa^n - \mathrm{Id} = \mathsf{b}\circ\kappa^n\circ\mathsf{d}, \qquad \kappa^{n+1}\circ\mathsf{d} = \mathsf{d} \qquad \text{hold on } \Omega^n, \quad \forall n = 1,2,\ldots. \tag{5.1.1}$$

The Karoubi operator κ commutes with b, and hence induces a well-defined endomorphism of the vector space $\Omega^n/\mathsf{b}\Omega^n$, $n = 1, 2, \ldots$. Furthermore, from the first identity in (5.1.1) we see that $\kappa^n = \mathrm{Id}$ on $\Omega^n/\mathsf{b}\Omega^n$. Hence, we have a direct sum decomposition $\Omega_\natural = (\Omega_\natural)^\kappa \oplus (\mathrm{Id} - \kappa)\Omega_\natural$. It follows that the natural projection $\Omega_\natural = \Omega/\mathsf{b}\Omega \twoheadrightarrow \mathrm{DR}^\bullet A = \Omega_\natural/(\mathrm{Id}-\kappa)\Omega_\natural$ restricts to an isomorphism $(\Omega_\natural)^\kappa \xrightarrow{\sim} \mathrm{DR}^\bullet A$. Parts (ii)–(iii) of Lemma 4.4.2 are clear from the proof of [Lo], Lemma 2.6.8. $\square$

Proof of Proposition 4.4.1. The first statement of the Proposition is immediate from the formula of Lemma 3.4.1(i). To prove the second statement, we exploit the first identity in (5.1.1). Using the formula for $\imath_\Delta$ and the fact that b commutes with κ, we compute that

$$(\kappa - 1) \circ \imath_\Delta = \mathsf{b} \circ (\kappa - 1) \circ (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) = \mathsf{b} \circ (\kappa^n - 1) = \mathsf{b}^2 \circ \kappa^n \circ \mathsf{d} = 0. \tag{5.1.2}$$

Hence, we deduce that the image of $\imath_\Delta$ is contained in $(\mathsf{b}\Omega)^\kappa$. Conversely, given any element $\alpha = \mathsf{b}(\beta) \in (\mathsf{b}\Omega)^\kappa$, we find that

$$\imath_\Delta(\beta) = (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ \mathsf{b}\beta = n \cdot \mathsf{b}\beta = n \cdot \alpha.$$

Thus, we have $\mathrm{Im}(\imath_\Delta) = (\mathsf{b}\Omega)^\kappa = (\,[A, \Omega]\,)^\kappa$, since $\mathsf{b}\Omega = [A, \Omega]$. Furthermore, it is clear that the two maps $(1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ \mathsf{b}$ and b coincide on $(\Omega^\bullet)^\kappa_\natural$, and hence have the same kernel. The exact sequence of the proposition now follows from Lemma 4.4.2. $\square$

5.2. **Harmonic decomposition.** Our proof of Theorem 4.2.2 is an adaptation of the strategy used in [CQ2, §2], based on the *harmonic decomposition*

$$\overline{\Omega} = P\overline{\Omega} \oplus P^\perp\overline{\Omega}, \quad \text{where} \quad P\overline{\Omega} := \mathrm{Ker}(\mathrm{Id} - \kappa)^2, \quad P^\perp\overline{\Omega} := \mathrm{Im}(\mathrm{Id} - \kappa)^2. \tag{5.2.1}$$

The differentials B, b, and d commute with κ, hence preserve the harmonic decomposition. Moreover, the differentials B and d are known to be proportional on $P\overline{\Omega}$. Specifically, introduce two degree preserving linear maps N, $\mathsf{N!} :\ \overline{\Omega} \to \overline{\Omega}$, such that, for any $n \geq 0$,

$$\mathsf{N}|_{\overline{\Omega}^n} \quad \text{is multiplication by } n, \quad \text{and} \quad \mathsf{N!}|_{\overline{\Omega}^n} \quad \text{is multiplication by } n!. \tag{5.2.2}$$

Then, exploiting the second identity in (5.1.1), it has been shown in [CQ2, §2, formula (11)] that

$$\mathsf{B}|_{P\overline{\Omega}} = (\mathsf{N} + 1) \cdot \mathsf{d}|_{P\overline{\Omega}}. \tag{5.2.3}$$

Next, we claim that

$$\text{(i)} \;\; \imath_\Delta|_{P^\perp\overline{\Omega}} = 0, \qquad \text{and} \qquad \text{(ii)} \;\; \imath_\Delta = \mathsf{N} \cdot \mathsf{b}. \tag{5.2.4}$$

To prove (i) we use that b commutes with κ. Therefore, applying (5.1.2), we find $i_\Delta \circ (\mathrm{Id} - \kappa)^2 = (\kappa - 1) \circ i_\Delta \circ (\kappa - 1) = 0$. To prove (ii), let $\alpha \in \overline{\Omega}^n$. From the first identity in (5.1.1), $\alpha - \kappa^n(\alpha) \in \mathsf{b}\overline{\Omega}$. Hence, $\mathsf{b}\alpha - \kappa^n(\mathsf{b}\alpha) \in \mathsf{b}^2\overline{\Omega} = 0$, since $\mathsf{b}^2 = 0$. Thus, the operator κ has finite order on $\mathsf{b}\overline{\Omega}$, and hence on $\mathsf{b}(P\overline{\Omega})$. But, for any operator T of finite order, $\mathrm{Ker}(\mathrm{Id} - T) = \mathrm{Ker}((\mathrm{Id} - T)^2)$. It follows that, if $\alpha \in P\overline{\Omega}^n$, then $\mathsf{b}\alpha \in \mathrm{Ker}((\mathrm{Id} - \kappa)^2) = \mathrm{Ker}(\mathrm{Id} - \kappa)$. We conclude that the element $\mathsf{b}\alpha$ is fixed by κ. Hence, $(1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ \mathsf{b}\alpha = n \cdot \mathsf{b}\alpha$. Therefore, by Proposition 4.4.1, $\imath_\Delta(\alpha) = n \cdot \mathsf{b}\alpha$, and (5.2.4) is proved. $\square$

5.3. **Proof of Theorem 4.2.2.** Since the harmonic decomposition is stable under all four differentials $\mathsf{B}, \mathsf{b}, \mathsf{d}$, and $\imath_\Delta$, we may analyze the homology of each of the direct summands, $P\overline{\Omega}$ and $P^\perp\overline{\Omega}$, separately.

First of all, it has been shown by Cuntz and Quillen [CQ2, Proposition 4.1(1)] that $\mathsf{B} = 0$ on $P^\perp\Omega$, and moreover that $(P^\perp\Omega, \mathsf{b})$ is acyclic.

Furthermore, since the complex $(\overline{\Omega}, \mathsf{d})$ is acyclic (see [CQ2, §1] or [CBEG] formula (2.5.1)), we deduce that

$$\textit{Each of the complexes } (P\overline{\Omega}, \mathsf{d}) \textit{ and } (P^\perp\overline{\Omega}, \mathsf{d}) \textit{ is acyclic.} \tag{5.3.1}$$

Now, the map $\imath_\Delta$ vanishes on $P^\perp\overline{\Omega}$ by (5.2.4)(i). Hence, on $P^\perp\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]$, we have $\mathsf{d}+t\cdot\imath_\Delta=\mathsf{d}$. Therefore, we conclude using (5.3.1) that $(P^\perp\overline{\Omega}[t],\mathsf{d})$, and hence also $(P^\perp\overline{\Omega}[t],\mathsf{d}+t\cdot\imath_\Delta)$, are acyclic complexes.

Thus, to complete the proof of the theorem, we must compare cohomology of the complexes $(P\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}],\mathsf{d}+t\cdot\imath_\Delta)$ and $(P\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}],\mathsf{B}+t\cdot\mathsf{b})$. We have $\mathsf{N}\cdot\mathsf{d}+(\mathsf{N}+1)^{-1}\cdot t\cdot\imath_\Delta=\mathsf{B}+t\mathsf{b}$. Post-composing this by $\mathsf{N}!$ (see (5.2.2)), we obtain $(\mathsf{N}!)\cdot(\mathsf{d}+t\cdot\imath_\Delta)=(\mathsf{B}+t\cdot\mathsf{b})\cdot(\mathsf{N}!)$. We deduce the following isomorphism of the complexes which completes the proof of the theorem:

$$\mathsf{N}!:\ (P\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}],\,\mathsf{d}+t\cdot\imath_\Delta)\ \xrightarrow{\sim}\ (P\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}],\,\mathsf{B}+t\cdot\mathsf{b}).\qquad\square$$

5.4. **Negative cyclic homology.** It is possible to extend Theorem 4.2.2 to the case of (nonperiodic) cyclic homology and negative cyclic homology using harmonic decomposition. To explain this, put

$$(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_+\ :=\ \bigoplus_{m\in\mathbb{Z}}\Big(\prod\nolimits_{i<m}\ t^i\Omega^{m-2i}\Big),\qquad (\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{\geq 0}\ :=\ \bigoplus_{m\in\mathbb{Z}}\Big(\prod\nolimits_{i\leq m}\ t^i\Omega^{m-2i}\Big).$$

It follows from definitions that cyclic homology and negative cyclic homology, respectively, may be defined in terms of the following complexes, cf. [Lo], ch. 2-3:

$$HC_\bullet=H^{-\bullet}\big(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]/(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_+,\,\mathsf{B}{+}t{\cdot}\mathsf{b}\big),\quad HC^-_\bullet=H^{-\bullet}\big((\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{\geq 0},\,\mathsf{B}{+}t{\cdot}\mathsf{b}\big).\quad(5.4.1)$$

Furthermore, we introduce two other homology theories, ${}^\heartsuit HC$, ${}^\heartsuit HC^-$. The corresponding homology groups are defined as the homology groups of complexes similar to (5.4.1), but where the differential $\mathsf{B}+t\cdot\mathsf{b}$ is replaced by $\mathsf{d}+t\cdot\imath_\Delta$.

Now, in terms of the projection to the harmonic part, cf. (5.2.1), we have

Proposition 5.4.2. *There are natural graded $\Bbbk[t]$-module isomorphisms*

$$P({}^\heartsuit HC_\bullet)\cong HC_\bullet\quad\text{and}\quad P({}^\heartsuit HC^-_\bullet)\cong HC^-_\bullet.$$

Proof. There are natural splittings

$$\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]=\frac{\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]}{(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_+}\bigoplus(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_+,\quad \overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]=\frac{\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]}{(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{\geq 0}}\bigoplus(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{\geq 0}.$$

These splittings are stable under the differential $t\cdot\mathsf{b}$. It follows that $P^\perp HC=0=P^\perp HC_-$, since $\mathsf{B}=0$ on $P^\perp\overline{\Omega}$, and the differential $t\cdot\mathsf{b}$ is acyclic here.

Observe further that, while $\imath_\Delta$ is zero on $P^\perp\overline{\Omega}$, the above splittings do *not* stabilize d. So, we can pick up some nonzero groups $P^\perp({}^\heartsuit HC)$, or $P^\perp({}^\heartsuit HC^-)$. However, restricting to the harmonic part, the proof of Theorem 4.2.2 yields an isomorphism of complexes

$$\big(P[\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]/(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_+],\ \mathsf{B}+t\cdot\mathsf{b}\big)\ \cong\ \big(P[\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}]/(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_+],\ \mathsf{d}+t\cdot\imath_\Delta\big),$$

and also a similar isomorphism involving $(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}])_{\geq 0}$. $\square$

5.5. **Proof of Theorem 4.3.1.** Let $B\subset A$ and assume that A/B is a free B-module.

Lemma 5.5.1. *Let $\{a_s,\ s\in\mathscr{S}\}$ be a basis of A/B as a free B-module. Then, the elements below form a basis for $\Omega^B(A)$ as a free $\Omega^\bullet(B)$-module:*

$$a_{s_0}\mathsf{d}a_{s_1}\mathsf{d}a_{s_2}\cdots\mathsf{d}a_{s_m},\quad \mathsf{d}a_{s_1}\mathsf{d}a_{s_2}\cdots\mathsf{d}a_{s_m},\quad s_j\in\mathscr{S}.\qquad(5.5.2)$$

Proof. Observe that the short exact sequence $B\to A\to A/B$ splits as a sequence of B-modules, since A/B is a free B-module. Hence, the elements 1 and $\{a_s,\ s\in\mathscr{S}\}$ give a B-module basis of A.

Let $\{b_r,\ r\in\mathscr{R}\}$ be a $\Bbbk$-basis of B including the element 1. Thus, $\mathscr{S}':=\mathscr{R}\times\mathscr{S}$ forms a $\Bbbk$-basis of A/B, by the assignment $(r,s)\mapsto a'_{r,s}:=b_ra_s$. We will apply the *diamond lemma*, see e.g., [Ber],

[Sch, §A.2], to show that (5.5.2) is indeed a $\Omega^\bullet B$-basis of $\Omega^B A$. This is fairly straightforward, but requires some formal details.

We introduce a partial order on the set of monomials in the alphabet $\mathsf{d}a'_{s'}, a_s \mathsf{d}a'_{s'}, s \in \mathscr{S}, s' \in \mathscr{S}'$ (with coefficients in $\Omega^\bullet B$) as follows. We say that $\alpha \prec \beta$ for monomials α, β in $A, \mathsf{d}A$ if either of the following alternatives holds:

(1) α has lower degree than β, i.e., fewer elements $\mathsf{d}a'_{s'}$;
(2) α and β have equal degrees and the last occurrence of an element from A (rather than $\mathsf{d}A$) in α occurs before the last occurrence of an element from A in β;
(3) the conditions in (1) and (2) are the same, but the last occurrence of a term $\mathsf{d}a$ in α where a is not a basis element of A/B occurs before the last such occurrence in β.
(4) If all of the above conditions are the same, then we order monomials using the lexicographical ordering induced by orderings of $\mathscr{R}, \mathscr{S}$ such that, in $\mathscr{R}$, 1 comes first, giving $\mathscr{S}' = \mathscr{R} \times \mathscr{S}$ itself the lexicographical ordering.

We now apply the Diamond Lemma for free modules over B [Ber], [Sch, §A.2]. The **reductions**, i.e., application of relations which lower the order of monomials appearing with nonzero coefficients, are of the form

$$(\mathsf{d}a'_{s'_1})a'_{s'_2} = \mathsf{d}(a'_{s'_1}a'_{s'_2}) - a'_{s'_1}\mathsf{d}a'_{s'_2}, \qquad s'_1, s'_2 \in \mathscr{S}', \tag{5.5.3}$$

and, for each $r \in \mathscr{R}, s' \in \mathscr{S}'$, one of

$$\mathsf{d}(b_r \mathsf{d}a'_{s'}) = b_r \mathsf{d}a'_{s'} + (\mathsf{d}b_r)a'_{s'}, \quad \text{or} \quad b_r \mathsf{d}a'_{s'} = \mathsf{d}(b_r a'_{s'}) - (\mathsf{d}b_r)a'_{s'}. \tag{5.5.4}$$

These reductions in particular generate the kernel of the quotient $T_B(A \oplus (A \otimes (\mathsf{d}A) \otimes A)) \twoheadrightarrow \Omega^B A$, which takes the free module spanned by our monomials to the desired quotient $\Omega^B A$. So, our result follows once we demonstrate that, whenever two different reductions are possible, then the results of both reductions have a common reduction. We briefly (and somewhat informally) explain how to prove this in the following paragraphs.

If the two relations that can be applied are both of type (5.5.3), then this is equivalent to the well-known fact that $\Omega^\bullet A \cong A \otimes (A/\Bbbk)^{\otimes \bullet}$. It also is easy to check directly: the only difficulty is the case where the $\mathsf{d}a'_{s'}$ term involved in both reductions is the same, i.e.,

$$(\mathsf{d}a'_{s'_1})a'_{s'_2}a'_{s'_3} = (\mathsf{d}a'_{s'_1})\sum\nolimits_{s'} \lambda_{s'}a'_{s'} + (\mathsf{d}a'_{s'_1})\lambda,$$

and here the reduction on the right yields the same result as the reduction on the left (pulling the $a'_{s'_2}$ under the differential) followed by a second reduction (for the element $a'_{s'_3}$).

If the two relations that can be applied are both of type (5.5.4), then the statement follows because both have the common reduction in which the element(s) $\mathsf{d}a', \mathsf{d}a''$ involved in the reductions are re-expressed by writing a', a'' as a B-linear combination of elements $a'_{s'}$, and pulling out all of the B-coefficients using (5.5.4).

Finally, if a monomial can be reduced using either (5.5.3) or (5.5.4), then the only difficulty would be the case where the $\mathsf{d}a'$ term involved in both reductions is the same. Then, the claim follows since the rightmost terms of the following two chains of maps are identical (we omit subscripts for readability):

$$\begin{aligned} \mathsf{d}(ba')a'' &\longmapsto (\mathsf{d}(ba'a'') - ba'\mathsf{d}(a'')) \longmapsto (b\mathsf{d}(a'a'') + (\mathsf{d}b)a'a'' - ba'\mathsf{d}(a'')) \\ &\qquad\qquad = b\mathsf{d}(a'a'') - ba'\mathsf{d}(a'') + (\mathsf{d}b)a'a'', \\ \mathsf{d}(ba')a'' &\longmapsto (b\mathsf{d}(a')a'' + (\mathsf{d}b)a'a'') \longmapsto (b\mathsf{d}(a'a'') - ba'\mathsf{d}a'' + (\mathsf{d}b)a'a''). \end{aligned}$$

The above may be easily modified to deal with the case that $ba' \prec a'$ using the reduction $b(\mathsf{d}a') \mapsto \mathsf{d}(ba') - (\mathsf{d}b)a'$, and similarly, if $ba'a'' \prec a'a''$, using the reduction $b\mathsf{d}(a'a'') \mapsto \mathsf{d}(ba'a'') - (\mathsf{d}b)a'a''$. $\square$

It is instructive, for the proof of Theorem 4.3.1 presented below, to have in mind the situation of Remark 4.3.6, where $B = \Bbbk[\![x_1, \ldots, x_n]\!]$ and $A = (A_0[\![x_1, \ldots, x_n]\!],\ \star)$. Then, $\{a_s\}$ is a $\Bbbk$-basis of A_0, and the (topologically-free version of the) Lemma 5.5.1 becomes more obvious.

Remark 5.5.5. More generally, given an arbitrary *regular* commutative algebra B and a maximal ideal $\mathfrak{m} \subset B$, taking the $\mathfrak{m}$-adic completions $\widehat{A}_{\mathfrak{m}}$ and $\widehat{B}_{\mathfrak{m}}$ reduces to the above situation. $\Diamond$

Completion of the proof of Theorem 4.3.1. We take B as a ground ring and apply Theorem 4.2.2 (which applies over any commutative base ring containing $\Bbbk$). We deduce that

$$H^{-i}\big(\overline{\Omega}\hat{\otimes}\Bbbk[t,t^{-1}](A;B)\hat{\otimes}\Bbbk[t,t^{-1}],\ \mathsf{d} + t\imath_\Delta\big) \ \cong\ HP^B_\bullet(A), \qquad \forall i \in \mathbb{Z}.$$

Now Lemma 5.5.1 implies that the map in (4.3.3) is an isomorphism, since the basis for $\Omega^B A$ as a free $\Omega^\bullet(B)$-module is also a basis for the associated graded $\mathrm{gr}^i_F\,\Omega^B A$, and a B-module basis for $\Omega^\bullet(A;B) = \mathrm{gr}^0_F\,\Omega^B A$. The construction of the Gauss-Manin connection given in §4.3 completes the proof of the theorem. $\Box$

Proof of Corollary 4.3.7. We only need to show that, in the present setting, the map in (4.3.3) is an isomorphism. For that, we observe that the argument used in the proof goes through provided that A is only *topologically* free over B, and our claim follows. $\Box$

6. The Representation functor

6.1. **Evaluation map.** We fix a finite-dimensional $\Bbbk$-vector space V. Set $\mathrm{End} := \mathrm{End}_\Bbbk(V)$. For any affine schemes X, S, let $X(S) = \mathrm{Hom}(S, X) = \mathrm{Hom}_{\Bbbk\text{-alg}}(\Bbbk[X], \Bbbk[S])$ denote the S-points of X.

Given an algebra A, we may consider the set $\mathrm{Hom}_{\Bbbk\text{-alg}}(A, \mathrm{End})$ of all algebra maps $\rho : A \to \mathrm{End}$. More precisely, to any *finitely presented* associative $\Bbbk$-algebra A we associate an affine scheme of finite type over $\Bbbk$, to be denoted $\mathrm{Rep}(A,V)$, such that $\mathrm{Rep}(A,V)(B) \cong \mathrm{Hom}_{\Bbbk\text{-alg}}(A, B\otimes\mathrm{End})$. That is, the B-points of $\mathrm{Rep}(A,V)$ correspond to families of representations of A parameterized by $\mathrm{Spec}\,B$. Write $\Bbbk[\mathrm{Rep}(A,V)]$ for the coordinate ring of the affine scheme $\mathrm{Rep}(A,V)$, which will be always assumed to be *non-empty.*

The tensor product $\mathrm{End}\otimes\Bbbk[\mathrm{Rep}(A,V)]$ is an associative algebra of polynomial maps $\mathrm{Rep}(A,V) \to \mathrm{End}$. To each element $a \in A$, we associate the element $\widehat{a} \in \mathrm{End}\otimes\Bbbk[\mathrm{Rep}(A,V)]$, which on the level of points, has the form $\widehat{a} : \mathrm{Rep}(A,V)(B) \to \mathrm{End}\otimes B,\ \widehat{a}(\rho) = \rho(a)$. This yields an algebra homomorphism, called the *evaluation map* $\mathrm{ev} :\ A \longrightarrow \mathrm{End}\otimes\Bbbk[\mathrm{Rep}(A,V)],\ a \mapsto \widehat{a}$.

6.2. **Extended de Rham complex and equivariant cohomology.** Let X be an arbitrary scheme with structure sheaf $\mathcal{O}_X$, tangent sheaf $\mathscr{T}_X := \mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X)$, and sheaf of Kähler differentials Ω^1_X. We write $\mathscr{T}(X)$ for the Lie algebra of global sections of the sheaf $\mathscr{T}_X$, and $\Omega^\bullet(X) = \Gamma(X,\ \wedge^\bullet_{\mathcal{O}_X}\Omega^1_X)$ for the DG algebra of differential forms, equipped with the de Rham differential.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $\mathfrak{g}$ act on $\Bbbk[\mathfrak{g}]$, the polynomial algebra on the vector space $\mathfrak{g}$, by the adjoint action. We view $\Bbbk[\mathfrak{g}]$ as an even-graded algebra such that the vector space of linear functions on $\mathfrak{g}$ is assigned degree 2.

Given a Lie algebra map $\mathfrak{g} \to \mathscr{T}(X),\ e \mapsto \vec{e}$, we get a $\mathfrak{g}$-action $\omega \mapsto L_{\vec{e}}\,\omega$ on $\Omega^\bullet(X)$, by the Lie derivative. This makes the tensor product $\Omega^\bullet(X, \mathfrak{g}) := \Omega^\bullet(X) \otimes \Bbbk[\mathfrak{g}]$ a graded algebra, equipped with the total grading and with the $\mathfrak{g}$-diagonal action. Let $i_{\vec{e}}$ denote the contraction. Then, set

$$d_\mathfrak{g} :\ \Omega^\bullet(X,\,\mathfrak{g}) \longrightarrow \Omega^{\bullet+1}(X,\,\mathfrak{g}), \quad \omega\otimes f \longmapsto \sum\nolimits_{r=1}^{\dim\mathfrak{g}} (i_{\vec{e}_r}\omega)\otimes(e^*_r\cdot f), \tag{6.2.1}$$

where $\{e_r\}$ and $\{e^*_r\}$ stand for dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. This map restricts to a differential $d_\mathfrak{g}$ on $\Omega^\bullet(X,\,\mathfrak{g})^\mathfrak{g}$, the graded subalgebra of $\mathfrak{g}$-diagonal invariants.

Definition 6.2.2. A differential form $\omega \in \Omega^\bullet(X)$ is called *basic* if, for any $e \in \mathfrak{g}$, both $L_{\vec{e}}\omega = 0$ and $i_{\vec{e}}\omega = 0$. Basic forms form a subcomplex $\Omega^\bullet_{\text{basic}}(X) \subset \Omega^\bullet(X)$ of the de Rham complex.

Furthermore, define the $\mathfrak{g}$-*equivariant algebraic de Rham complex* of X to be the complex

$$\big(\Omega^\bullet(X,\,\mathfrak{g})^{\mathfrak{g}},\, d_{\mathrm{DR}} + d_{\mathfrak{g}}\big), \qquad d_{\mathrm{DR}} := d \otimes \mathrm{id}_{\Bbbk[\mathfrak{g}]}. \tag{6.2.3}$$

We now return to the setup of §6.1. Thus we fix a finitely-presented algebra A, a finite-dimensional vector space V, and consider the scheme $\mathrm{Rep}(A, V)$.

Let $G = \mathrm{GL}(V)$. This is an algebraic group over $\Bbbk$ that acts naturally on the algebra End by inner automorphisms, via conjugation. Hence, given an algebra homomorphism $\rho : A \to \mathrm{End}$ and $g \in G(\Bbbk)$, we may define a conjugate homomorphism $g(\rho) : a \mapsto g \cdot \rho(a) \cdot g^{-1}$. Then, the action $\rho \mapsto g(\rho)$ makes $\mathrm{Rep}(A, V)$ a G-scheme (extending in the obvious way to B-valued representations for any B).

Let $\mathfrak{g} := \mathrm{Lie}\, G$ be the Lie algebra of G. The action of G on $\mathrm{Rep}(A, V)$ induces a Lie algebra map

$$\mathrm{act}_A :\ \mathfrak{g} \longrightarrow \mathscr{T}(\mathrm{Rep}(A, V)), \quad x \longmapsto \vec{x} = \mathrm{act}_A(x). \tag{6.2.4}$$

Thus, we may consider $\Omega^\bullet(\mathrm{Rep}(A, V),\, \mathfrak{g})^{\mathfrak{g}}$, the corresponding $\mathfrak{g}$-equivariant algebraic de Rham complex.

Now, thanks to Lemma 3.4.1(ii), the map $\mathsf{d} + i_\Delta : \mathrm{DR}_t A \to \mathrm{DR}_t A$, squares to zero. We call the resulting complex $\big(\mathrm{DR}_t A,\ \mathsf{d} + i_\Delta\big)$ the $\textsf{noncommutative equivariant de Rham complex}$. The first isomorphism of the following theorem, which is the main result of this section, shows that this complex is indeed a noncommutative analogue of the equivariant de Rham differential (6.2.3).

Recall the operator N from (5.2.2).

Theorem 6.2.5. *The evaluation map induces the following canonical morphisms of complexes:*

$$\big(H_\bullet(A, A),\ \mathsf{B}\big) \xrightarrow{\ \mathrm{ev}\ } \big(\Omega^\bullet_{basic}(\mathrm{Rep}(A, V)),\ (\mathsf{N}+1)\circ d_{DR}\big), \quad \text{and} \tag{6.2.6}$$

$$\big(\mathrm{DR}^\bullet_t A,\, \mathsf{d} + i_\Delta\big) \xrightarrow{\ \mathrm{ev}\ } \big(\Omega^\bullet(\mathrm{Rep}(A, V),\, \mathfrak{g})^{\mathfrak{g}},\ d_{DR} + d_{\mathfrak{g}}\big). \tag{6.2.7}$$

We begin the proof with some general constructions.

6.3. **Evaluation map on differential forms.** Observe that giving an algebra homomorphism $\rho : \Bbbk[t] \to B \otimes \mathrm{End}$ amounts to specifying an arbitrary element $x = \rho(t) \in B \otimes \mathrm{End}$. Thus, we have $\mathrm{Rep}(\Bbbk[t], V)(B) = B \otimes \mathrm{End}$.

Similarly, for any algebra A, giving an algebra morphism $\rho : A * \Bbbk[t] \to B \otimes \mathrm{End}$ amounts to giving a homomorphism $A \to B \otimes \mathrm{End}$ and an arbitrary additional element $x = \rho(t) \in B \otimes \mathrm{End}$. We see that $\mathrm{Rep}(A_t, V) \cong \mathrm{Rep}(A, V) \times \mathrm{Rep}(\Bbbk[t], V)$. Let π denote the second projection, which is G-equivariant. We will use shorthand notation

$$\mathrm{Rep}_t := \mathrm{Rep}(A_t, V) = \mathrm{Rep}(A, V) \times \mathrm{Rep}(\Bbbk[t], V), \quad \text{and} \quad \mathrm{Rep} := \mathrm{Rep}(A, V).$$

Let $\Omega^\bullet_\pi(\mathrm{Rep}_t)$ be the DG algebra of relative (with respect to π) algebraic differential forms on the scheme Rep_t (in the ordinary sense of commutative algebraic geometry). By definition, we have

$$\Omega^\bullet_\pi(\mathrm{Rep}_t) := \mathsf{\Lambda}^\bullet_{\Bbbk[\mathrm{Rep}_t]}\, \Omega^1_\pi(\mathrm{Rep}_t) \cong \Omega^\bullet(\mathrm{Rep}) \otimes \Bbbk[\mathrm{Rep}(\Bbbk[t], V)]. \tag{6.3.1}$$

Generalizing the construction of §6.1, we now introduce an evaluation map on relative differential forms. In more detail, given $n = 0, 1, 2, \ldots$, write $m : \mathrm{End}^{\otimes(n+1)} \to \mathrm{End}$ for the n-fold multiplication map. We define a map ev_Ω as the following composite:

$$\begin{aligned}\Omega^n_{\Bbbk[t]}(A_t) = A_t \otimes (A_t/\Bbbk[t])^{\otimes n} &\overset{\mathrm{ev}}{\to} (\mathrm{End}\otimes\Bbbk[\mathrm{Rep}_t]) \otimes \big(\mathrm{End}\otimes\Omega^1_\pi(\mathrm{Rep}_t)\big)^{\otimes n} \\ &\to \mathrm{End}^{\otimes n+1} \otimes \bigotimes\big(\mathsf{\Lambda}^n_{\Bbbk[\mathrm{Rep}_t]}\, \Omega^1_\pi(\mathrm{Rep}_t)\big) \xrightarrow{\ m\otimes\mathrm{Id}\ } \mathrm{End}\otimes\Omega^n_\pi(\mathrm{Rep}_t).\end{aligned}$$

Any element in the image of this composite is easily seen to be G-invariant with respect to the G-diagonal action on $\mathrm{End}\otimes\Omega^n_\pi(\mathrm{Rep}_t)$. Thus, the composite above yields a well-defined, canonical DG algebra map

$$\mathrm{ev}_\Omega : \Omega_t A \to \big(\mathrm{End}\otimes\Omega^\bullet_\pi(\mathrm{Rep}_t)\big)^G, \quad \alpha = a_0\, \mathsf{d}a_1 \ldots \mathsf{d}a_n \mapsto \widehat{\alpha} = \widehat{a}_0\ d_{\mathrm{DR}}\widehat{a}_1 \ldots d_{\mathrm{DR}}\widehat{a}_n.$$

Furthermore, we have the linear function $\mathrm{Tr} : \mathrm{End} \to \Bbbk$, $x \mapsto \mathrm{Tr}(x)$. We form the composite

$$\Omega_t A \xrightarrow{\mathrm{ev}_\Omega} \big(\mathrm{End}\otimes\Omega^\bullet_\pi(\mathrm{Rep}_t)\big)^G \xrightarrow{\mathrm{Tr}\otimes\mathrm{Id}} \big(\Bbbk\otimes\Omega^\bullet_\pi(\mathrm{Rep}_t)\big)^G = \Omega^\bullet_\pi(\mathrm{Rep}_t)^G, \quad \alpha \mapsto \mathrm{Tr}\,\widehat{\alpha}. \tag{6.3.2}$$

The above composite clearly commutes with the de Rham differentials. Furthermore, it vanishes on the (graded) commutator space $[\Omega_t A, \Omega_t] \subset \Omega_t A$, due to symmetry of the trace function. Therefore, the map in (6.3.2) descends to $\mathrm{DR}^\bullet(\Omega_t A)$.

We remark next that the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ is nothing but the associative algebra End viewed as a Lie algebra. Hence, using the isomorphisms in (6.3.1), we can write

$$\Omega^\bullet_\pi(\mathrm{Rep}_t) = \Omega^\bullet(\mathrm{Rep})\otimes\Bbbk[\mathrm{Rep}(\Bbbk[t], V)] = \Omega^\bullet(\mathrm{Rep})\otimes\Bbbk[\mathfrak{g}] = \Omega^\bullet(\mathrm{Rep},\, \mathfrak{g}).$$

Thus, by the definition of the extended de Rham complex, $\mathrm{DR}^\bullet_t A$, the composite in (6.3.2) gives a map

$$(\mathrm{Id}\otimes\mathrm{Tr})\circ\mathrm{ev}_\Omega :\ \mathrm{DR}^\bullet_t A \longrightarrow \Omega^\bullet_\pi(\mathrm{Rep}_t)^G = \Omega^\bullet(\mathrm{Rep},\, \mathfrak{g})^{\mathfrak{g}}. \tag{6.3.3}$$

6.4. **Proof of Theorem 6.2.5.** It is clear that d is clearly taken to d_{DR} under (6.3.3). Hence, proving (6.2.7), where the map 'ev' stands for $(\mathrm{Id}\otimes\mathrm{Tr})\circ\mathrm{ev}_\Omega$, amounts to showing commutativity of the diagram

$$\begin{tikzcd} \mathrm{DR}_t A \arrow[r, "(\mathrm{Id}\otimes\mathrm{Tr})\circ\mathrm{ev}_\Omega"] \arrow[d, "i_\Delta"'] & \Omega(\mathrm{Rep},\, \mathfrak{g})^{\mathfrak{g}} \arrow[d, "d_{\mathfrak{g}}"] \\ \mathrm{DR}_t A \arrow[r, "(\mathrm{Id}\otimes\mathrm{Tr})\circ\mathrm{ev}_\Omega"] & \Omega(\mathrm{Rep},\, \mathfrak{g})^{\mathfrak{g}}. \end{tikzcd} \tag{6.4.1}$$

To see this, we note that, for any $a_0, \ldots, a_n \in A_t$,

$$\mathrm{ev}_\Omega \circ i_\Delta[a_0\, \mathsf{d}a_1\, \mathsf{d}a_2 \ldots \mathsf{d}a_n] = (\mathrm{Id}\otimes\mathrm{Tr})\Big(\sum_{i=1}^n \widehat{a}_0\ d_{\mathrm{DR}}\widehat{a}_1 \ldots d_{\mathrm{DR}}\widehat{a}_{i-1}\ [\widehat{t}, \widehat{a}_i]\ d_{\mathrm{DR}}\widehat{a}_{i+1} \ldots d_{\mathrm{DR}}\widehat{a}_n\Big). \tag{6.4.2}$$

Next, note that $\hat{t}$ may be identified with the element $\mathrm{Id} \in \mathfrak{g}\otimes\mathfrak{g}^* \cong \mathrm{End}\otimes\mathfrak{g}^* \subset \mathrm{End}\otimes\Bbbk[\mathfrak{g}]$. Furthermore, for any element $e \in \mathfrak{g}$ and any $s \in \Bbbk[\mathrm{Rep}(A, V)]\otimes\mathrm{End}$, we evidently have $i_{\vec{e}}(ds) = \mathrm{ad}\, e(s) = [(1\otimes e), s]$. As a consequence, by (6.2.1), we obtain that the RHS of (6.4.2) may be identified with $d_{\mathfrak{g}}((\mathrm{Id}\otimes\mathrm{Tr})(\widehat{a}_0\mathsf{d}\widehat{a}_1 \ldots \mathsf{d}\widehat{a}_n))$, as desired.

To prove (6.2.6), ev becomes the restriction of $(\mathrm{Id}\otimes\mathrm{Tr})\circ\mathrm{ev}_\Omega$ to $\ker(i_\Delta) \subset \Omega^\bullet(\mathrm{Rep})^{\mathfrak{g}}$ (recall Theorem 4.1.1). Commutativity of (6.4.1) together with Theorem 4.1.1 immediately gives that this induces a morphism $H_\bullet(A, A) \to \Omega^\bullet_{basic}(\mathrm{Rep}(A, V))$. It remains only to show that B is carried to $(\mathsf{N}+1)\mathsf{d}$. To see this, we use the harmonic decomposition (5.2.1). Under the quotient $\overline{\Omega}^\bullet A \twoheadrightarrow \mathrm{DR}^\bullet A/\Bbbk$ (considering $\Bbbk$ to be the span of the image of $1 \in A$), $P^\perp\overline{\Omega}$ is killed, so the differential B is carried to $(\mathsf{N}+1)d_{\mathrm{DR}}$. Thus, on $\mathrm{DR}^\bullet A$, the differential B must reduce to the same as $(\mathsf{N}+1)\mathsf{d}$ up to a scalar. However, since B has degree $+1$, the scalar must be zero. So $\mathsf{B} = (\mathsf{N}+1)\mathsf{d}$ on $\mathrm{DR}^\bullet A$. Thus, the same is true after passing to $\ker(i_\Delta)$.

7. Free products and deformations

7.1. **First order deformations based on free products.** Recall that, given an associative algebra A, we let $A_t = A * \Bbbk[t]$ and write $I = A_t^+ = (t) \subset A_t$ for the augmentation ideal.

A *first order free product deformation* of an associative algebra A is the structure of an associative algebra on the vector space A_t/I^2 that makes the vector space $I/I^2 \subset A_t/I^2$ a two-sided ideal and that makes the natural bijection below an algebra isomorphism,

$$\big(A_t/I^2\big)/(I/I^2) = A_t/I \ \xrightarrow{\sim}\ A.$$

It is convenient to identify the vector space A_t/I^2 with $A \oplus (A \otimes A)$, using (1.3.2). Thus, we are interested in associative products on the vector space $A \oplus (A \otimes A)$ that have the following form:

$$\big(u \oplus (u' \otimes u'')\big) \ \times\ \big(v \oplus (v' \otimes v'')\big) \ \overset{\star_\beta}{\longmapsto}\ uv \oplus \big(u' \otimes u'' v + uv' \otimes v'' + \beta(u,v)\big), \tag{7.1.1}$$

where $\beta : A \times A \to A \otimes A$ is a certain $\Bbbk$-bilinear map.

These products are taken up to an equivalence. Specifically, for any $\Bbbk$-linear map $f : A \to A \otimes A$, we define a linear bijection

$$\tilde f :\ A \oplus (A \otimes A) \longrightarrow A \oplus (A \otimes A), \quad u \oplus (u' \otimes u'') \longmapsto u \oplus (u' \otimes u'' + f(u)).$$

Given a product $\star_\beta$ and a map f, we define a new product by transporting the structure via $\tilde f$, that is, by the formula $x \star_\gamma y := \tilde f^{-1}(\tilde f(x) \star_\beta \tilde f(y))$. We say that the products $\star_\gamma$ and $\star_\beta$ are *equivalent*.

A routine calculation, completely analogous to the classical one due to Gerstenhaber, now yields:

- A first product $\star_\beta$ as in (7.1.1) is associative $\iff \beta \in C^2(A, A \otimes A)$ is a Hochschild 2-cocycle with coefficients in $A \otimes A$.
- The products $\star_\beta$ and $\star_\gamma$ corresponding to 2-cochains β and γ are equivalent $\iff \beta - \gamma$ is a Hochschild coboundary.

Thus, similarly to the classical theory, we obtain a classification of first order star product deformations in terms of Hochschild cohomology as follows

Proposition 7.1.2. *Equivalence classes of associative products, as in* (7.1.1)*, are in one-to-one correspondence with the elements of $H^2(A, A \otimes A)$, the second Hochschild cohomology group of the A-bimodule $A \otimes A$.* $\square$

To study *higher order* free product deformations, we have to introduce first some new operations on Hochschild cohomology, to be defined below.

7.2. For any algebra A, the natural imbedding $A \hookrightarrow A_t$ makes A_t a graded A-bimodule. Using the identification (2.3.1), we may write $A_t = \bigoplus_{k \geq 1} A^{\otimes k}$. Here, the summand $A^{\otimes k}$ is assigned grade degree $2k-2$ and is equipped with the *outer* A-bimodule structure defined by the formula $b(a' \otimes u \otimes a'')c := (ba') \otimes u \otimes (a''c)$, for any $a', a'', b, c \in A$ and $u \in A^{\otimes (k-2)}$.

Let $C^\bullet(A, A_t) = \bigoplus_{p,k \geq 1} C^p(A, A^{\otimes k})$ be the Hochschild cochain complex with coefficients in the A-bimodule A_t. Multiplication in the algebra A_t induces, for any $p, q, k, m \geq 1$, a cup-product

$$\cup :\ C^p(A, A^{\otimes k}) \times C^q(A, A^{\otimes m}) \longrightarrow C^{p+q}(A, A^{\otimes (k+m-1)}),$$

This way, $C^\bullet(A, A_t)$ acquires the structure of a bigraded associative algebra such that the direct summand $C^p(A, A^{\otimes k})$ is assigned bidegree $(p, 2k-2)$.

Next, on $C^\bullet(A, A_t)$, we introduce a pair of new binary operations, $\vdash$ and $\dashv$:

$$C^p(A, A^{\otimes k}) \times C^q(A, A^{\otimes m}) \longrightarrow C^{p+q-1}(A, A^{\otimes (k+m-1)}),$$
$$(f,g) \mapsto f \vdash g := f^{[1,p]} \circ g^{[p,p+q-1]}, \quad \text{and} \quad (f,g) \mapsto f \dashv g := g^{[k,k+q-1]} \circ f^{[1,p]},$$

where $f^{[i,j]}$ denotes applying f to the consecutive components $i, i+1, i+2, \ldots, j$, that is,

$$f^{[i,j]}(a_1 \otimes \ldots \otimes a_\ell) = a_1 \otimes \ldots \otimes a_{i-1} \otimes f(a_i \otimes \ldots \otimes a_j) \otimes a_{j+1} \otimes \ldots \otimes a_\ell.$$

Proposition 7.2.1. (i) *The operation* $f \vee g := f \vdash g \; - \; f \dashv g$ *and Hochschild differential* b *give the space* $C(A, A_t)_{\geq 2} := \bigoplus_{p,k \geq 2} C^p(A, A^{\otimes k})$ *the structure of an associative DG algebra, i.e.,*

$$f \vee (g \vee h) = (f \vee g) \vee h, \tag{7.2.2}$$

$$\mathsf{b}(f \vee g) = (\mathsf{b}f) \vee g + (-1)^{p-1} f \vee (\mathsf{b}g) \qquad \forall f \in C^p(A, A^{\otimes k}). \tag{7.2.3}$$

(ii) *The cup product* $\cup$ *is associative and induces the zero map on cohomology:*

$$H^p(A, A^{\otimes k}) \otimes H^q(A, A^{\otimes m}) \xrightarrow{\cup = 0} H^{p+q}(A, A^{\otimes k+m-1}) \quad \forall p, q \geq 1 \text{ whenever } k \geq 2 \text{ or } m \geq 2.$$

(iii) *We have the following compatibility identities:*

$$(f \cup g) \vee h = f \cup (g \vee h), \qquad f \vee (g \cup h) = (f \vee g) \cup h. \tag{7.2.4}$$

Proof. We note the following identities (for any x, y, z):

$$x \vdash (y \vdash z) = (x \vdash y) \vdash z, \quad x \dashv (y \dashv z) = (x \dashv y) \dashv z,$$
$$x \vdash (y \dashv z) = (x \vdash y) \dashv z, \quad x \dashv (y \vdash z) = (x \dashv y) \vdash z.$$

The first set of identities is fairly obvious from the definition, and the second follows because, since $y \in C^{p'}(A, A^{\otimes k'})$ for $p', k' \geq 2$, x (on the left) and z (on the right) are always applied to disjoint sets of consecutive components. This is all we need to prove the associativity (7.2.2). In fact, $\dashv$ and $\vdash$ are mutually associative (7.2.7).

To prove the DG property (7.2.3), we show the following two identities:

$$(\mathsf{b}f \vdash g) + (-1)^{p-1}(f \vdash \mathsf{b}g) = \mathsf{b}(f \vdash g) + (-1)^{p+1} f \cup g, \tag{7.2.5}$$

$$(\mathsf{b}f \dashv g) + (-1)^{p-1}(f \dashv \mathsf{b}g) = \mathsf{b}(f \dashv g) + (-1)^{p-1} f \cup g. \tag{7.2.6}$$

Actually, in (7.2.5), we only need $m \geq 2$, and in (7.2.6), we only need $k \geq 2$.

We show only (7.2.5), as the other identity is the same verification. Write $m : A \otimes A \to A$ for the multiplication map. We compute

$$((\mathsf{b}f \vdash g) + (-1)^{p-1}(f \vdash \mathsf{b}g))(a_1 \otimes \ldots \otimes a_{p+q}) = a_1 (f \vdash g)(a_2 \otimes \ldots \otimes a_{p+q})$$
$$+ (-1)^{p+q+1}(f \vdash g)(a_1 \otimes \ldots \otimes a_{p+q-1}) a_{p+q} + (-1)^{m+1} f \cup g + \sum_{i=1}^{p+q-1} (-1)^i (f \vdash g) \circ m^{i,i+1}.$$

Finally, part (iii) of the proposition follows from Proposition 7.2.8(ii) below. □

For each $n \geq 1$, we may introduce an operad, $As^{(n)}$, generated by n binary operations, $\star_i$, $i = 1, \ldots, n$, subject to the following relations of pairwise mutual associativity (considered also by J.-L. Loday):

$$a \star_i (b \star_j c) = (a \star_i b) \star_j c, \qquad \forall i, j = 1, \ldots, n. \tag{7.2.7}$$

Proposition 7.2.8. (i) *Each of the operads* $As^{(2)}$ *and* $As^{(3)}$ *is Koszul and self-dual, (see* [GK] *for a definition).*

(ii) *The operations* $(\vee, \cup)$ *and* $(\vdash, \dashv, \cup)$ *make* $C(A, A_t^+)_{\geq 2}$ *an* $As^{(2)}$*- and an* $As^{(3)}$*-algebra, respectively.*

Sketch of Proof. It is easy to see, as in the associative case, that the quadratic dual of $As^{(2)}$ is itself and similarly for $As^{(3)}$. Next we prove that our operads are Koszul. To see this, we use the same proof as in the associative case [GK], [MSS]. Specifically, we show that the operadic homology of the free $As^{(2)}$ or $As^{(3)}$-algebra vanishes in degrees ≥ 2. To this end, we split up the operadic homology complexes for $As^{(2)}$ and $As^{(3)}$ into direct sums of pieces corresponding to particular sequences of operations, e.g., $(*, \star)$ would consist of terms that multiply out to a sum of terms of the form $a * b \star c$. Each such has the vanishing homology property by the same proof as in the usual case of Hochschild homology of a free associative algebra; see [Lo, §3.1].

Next, it is straightforward to verify the following identities:

$$x \vdash (y \cup z) = (x \vdash y) \cup z, \quad x \dashv (y \cup z) = (x \dashv y) \cup z,$$
$$(x \cup y) \vdash z = x \cup (y \vdash z), \quad (x \cup y) \dashv z = x \cup (y \dashv z).$$

This yields part (ii), and also implies the identities in (7.2.4). □

7.3. **Infinite order deformations.** In the classical theory, an infinite order formal deformation of an algebra A with multiplication map $m : A \times A \to A$ is a formally associative star-product

$$a \star a' = m(a, a') + t\beta^{(1)}(a, a') + t^2\beta^{(2)}(a, a') + \ldots \in A[\![t]\!], \quad \beta^{(k)} \in C^2(A, A), \ \ k \geq 1. \qquad (7.3.1)$$

Given such a star-product, we extend the formal series $m + t\beta^{(1)} + t^2\beta^{(2)} + \ldots \in \sum t^k C^2(A, A)$ $= C^2(A, A[\![t]\!])$, by $\Bbbk[\![t]\!]$-bilinearity, to obtain a continuous cochain $\beta \in C^2_{\Bbbk[\![t]\!]}(A[\![t]\!], A[\![t]\!])$.

With an appropriate equivalence relation on the set of associative star-products, one has the following well-known result.

Proposition 7.3.2. *Equivalence classes of associative star products* (7.3.1) *are in one-to-one correspondence with gauge equivalence classes in the set of solutions of the following Maurer-Cartan equation*

$$\{\beta \in C^2_{\Bbbk[\![t]\!]}(A[\![t]\!], A[\![t]\!]) \ \mid \ t \cdot \mathsf{b}_{A[\![t]\!]}(\beta) + 1/2\{\beta, \beta\}_{A[\![t]\!]} = 0\}. \qquad \square$$

To consider free product deformations of an algebra A, let $\widehat{A}_t := \prod_{k \geq 0} A_t^{2k} \cong \prod_{m \geq 1} A^{\otimes m}$ be the completion of the free product algebra A_t in the t-adic topology, and write $\widehat{A}_t^+ := \prod_{k \geq 1} A_t^{2k}$ for the corresponding augmentation ideal.

An infinite order free product deformation of A is, by definition, a formally associative star-product of the form

$$a \star_\beta a' = aa' + \beta^{(1)}(a, a') + \beta^{(2)}(a, a') + \ldots, \qquad \beta^{(k)} \in C^2(A, A_t^{2k}). \qquad (7.3.3)$$

In more detail, given an arbitrary sequence $\beta^{(k)} \in C^2(A, A_t^{2k}), \ k = 1, 2, \ldots,$ of 2-cochains, we first extend each map $\beta^{(k)}$ to a $\Bbbk[t]$-bilinear map $\beta^{(k)} : A_t \times A_t \to A_t$ given by the formula

$$\beta^{(k)} : \ (a_1 t a_2 t \ldots t a_m) \times (b_1 t \ldots t b_n) \longmapsto a_1 t \ldots t a_{m-1} t \beta(a_m \otimes b_1) t b_2 t \ldots t b_n.$$

For any $u, u' \in A_t$, the corresponding formal series $uu' + \beta_t^{(1)}(u, u') + \beta_t^{(2)}(u, u') + \ldots$ clearly converges in $\widehat{A}_t$. In this way, we obtain a well defined and continuous $\Bbbk[t]$-bilinear map $A_t \times A_t \to \widehat{A}_t$, that can be uniquely extended, by continuity, to a map $\beta : \widehat{A}_t \times \widehat{A}_t \to \widehat{A}_t$. We are interested in those star-products (7.3.3) which give rise to an *associative* product β, on $\widehat{A}_t$.

We define a natural equivalence relation on such star-products as follows. Given a sequence of 1-cochains $f^{(k)} \in C^1(A, A_t^{2k})$, $k = 1, 2, \ldots,$ using (1.2.4), we construct similarly a continuous map $f = \mathrm{Id} + f_t^{(1)} + f_t^{(2)} + \ldots : \widehat{A}_t \to \widehat{A}_t$. Furthermore, given any star-product $\star_\beta$, we define a new star product by the formula $a \star_\gamma a' := f^{-1}(f(a) \star_\beta f(a'))$. We say that the star-products $\star_\gamma$ and $\star_\beta$ are *equivalent*.

Given a star-product (7.3.3), we form $\beta := m + \beta^{(1)} + \beta^{(2)} + \ldots \in C^2(A, \widehat{A}_t^+)$, an associated 2-cochain. We further define gauge equivalence of chains to be infinitesimally generated by the following $C^1(A, \widehat{A}_t^+)$-action: $\phi: \ \beta \mapsto \phi\cdot\beta$, where

$$\phi\cdot\beta(a_1\otimes a_2) = \phi(a_1)a_2 + a_1\phi(a_2) - \phi(a_1a_2) + \beta'(\phi(a_1)\otimes a_2) + \beta'(a_1\otimes\phi(a_2)) - \phi_t\circ\beta'(a_1\otimes a_2).$$

Here, ϕ_t is defined according to formula (1.2.4), and we put

$$\beta'(a_1t\ldots ta_m\otimes c_1t\ldots tc_n) = a_1t\ldots ta_{m-1}t(\beta(a_m\otimes c_1))tc_2t\ldots tc_n.$$

The following result provides a cohomological description of free product deformations, similar to the one given in Proposition 7.3.2 (below, b stands for the Hochschild differential).

Theorem 7.3.4. (i) *Linear maps $\beta^{(i)}$ in* (7.3.3) *define an associative product on $\widehat{A}_t$ iff the Maurer-Cartan equation,*

$$\mathsf{b}(\beta) + \frac{1}{2}\,\beta\vee\beta = 0, \qquad \text{holds for} \quad \beta := \beta^{(1)} + \beta^{(2)} + \ldots \in C^2(A, \widehat{A}_t^+). \tag{7.3.5}$$

(ii) *Star products are equivalent iff the solutions of the equation* (7.3.5) *are gauge-equivalent.*

Proof. In this proof (only) we temporarily change our notation and given $\beta^{(m)} \in C^2(A, A^{\otimes m+1})$, write $\beta_m^{12} \in C^3(A, A^{\otimes m+2})$ for the map $a\otimes b\otimes c \mapsto \beta^{(m)}(a,b)\otimes c$, etc.

It is easy to see that it suffices to check associativity on $A^{\otimes 3}$, namely that $(a\star b)\star c = a\star(b\star c)$ for all $a, b, c \in A$. This is equivalent to the following (similar to (7.1.1)), for all $p \geq 2$:

$$\beta_p(a\otimes bc) + a\beta_p(b\otimes c) + \sum_{m+n=p} \beta_m^{12}\circ\beta_n^{23}(a\otimes b\otimes c)$$
$$= \beta_p(ab\otimes c) + \beta_p(a\otimes b)c + \sum_{m+n=p} \beta_n^{m,m+1}\circ\beta_m^{12}(a\otimes b\otimes c). \tag{7.3.6}$$

which is the Maurer-Cartan equation (7.3.5). Part (ii) is then not difficult to verify. □

Below, we summarize a few basic properties of free product deformations which are entirely analogous to the well-known properties of ordinary 1-parameter formal deformations:

(1) First order free product deformations are classified by $H^2(A, A\otimes A)$.
(2) The obstruction to extending a first-order deformation to second order lives in $H^3(A, A^{\otimes 3})$.
(3) Let $\star_n$ be an associative product on $A/(A_t^+)^{n+1}$ of the form

$$a\star_n b = ab + \sum_{m=1}^{n} \beta^{(m)}(a\otimes b), \qquad \beta^{(m)} \in C^2(A, A^{m+1}) = C^2(A, A_t^{2m}). \tag{7.3.7}$$

There is an obstruction in $H^3(A, A^{\otimes n+1})$ to the existence of $\beta^{(n+1)} \in C^2(A, A^{\otimes(n+2)})$, such that the formula $a\star_{n+1} b := a\star_n b + \beta^{(n+1)}(a\otimes b)$ gives an associative product on $A/(t)^{n+2}$.

Explicitly, the condition on $\beta^{(n+1)}$ reads

$$\mathsf{b}\beta^{(n+1)} = \sum_{i+j-1=n+1} \beta^{(i)}\vee\beta^{(j)}. \tag{7.3.8}$$

(4) If the obstruction in (3) vanishes then, the space of possible $\beta^{(n+1)}$ (up to equivalence of the resulting star product, $\star + \beta^{(n+1)}$ modulo $(t)^{n+2}$), is $H^2(A, A^{\otimes(n+2)})$.

Proof. In degrees $n = 1, 2$, the Maurer-Cartan equation (7.3.8) says that

$$\mathsf{b}\beta^{(1)} = 0, \qquad \text{and} \qquad \mathsf{b}\beta^{(2)} = \beta^{(1)}\vee\beta^{(1)}.$$

Using (7.2.3), we find $\mathsf{b}(\beta^{(1)}\vee\beta^{(1)}) = (\mathsf{b}\beta^{(1)})\vee\beta^{(1)} + \beta^{(1)}\vee(\mathsf{b}\beta^{(1)}) = 0 + 0 = 0$. This yields (1)–(2).

In general, if we have $\beta^{(1)}, \ldots, \beta^{(n)}$ satisfying the Maurer-Cartan conditions up to $O(t^{n+1})$ (i.e., $\mathsf{b}\beta^{(m)} = \sum_{i+j-1=m} \beta^{(i)} \vee \beta^{(j)}$ for $m \leq n$), then we consider b of the RHS of (7.3.8):

$$\sum_{i+j-1=n+1} \mathsf{b}(\beta^{(i)} \vee \beta^{(j)}) = \sum_{i+j-1=n+1} \big[(\mathsf{b}\beta^{(i)}) \vee \beta^{(j)} - \beta^{(i)} \vee (\mathsf{b}\beta^{(j)})\big]$$
$$= \sum_{i+j+k-2=n+1} \big[(\beta^{(i)} \vee \beta^{(j)}) \vee \beta^{(k)} - \beta^{(i)} \vee (\beta^{(j)} \vee \beta^{(k)})\big] = 0,$$

where we have used both (7.2.3) and (7.2.2). Thus, the RHS is indeed a Hochschild 3-cocycle. Thus, if this represents the zero class of $H^3(A, A^{\otimes n+1})$ (i.e., it is a Hochschild 3-coboundary), then the space of choices of $\beta^{(n+1)}$ is the space of Hochschild 2-cocycles. Furthermore, we have the freedom of conjugating by automorphisms $\phi : A \to A$ of the form $\phi = \mathrm{Id} + \phi'$ as follows:

$$\phi^{-1}(\phi(a) \star \phi(b)) \equiv a \star b + \phi'(a)b + a\phi'(b) - \phi'(ab) \pmod{(t)^{n+2}}.$$

We conclude that the space of $\beta^{(n+1)}$'s, taken up to equivalence of the obtained star product on $A/(t)^{n+2}$, is $H^2(A, A^{\otimes n+2})$. $\square$

7.4. **Deformations of NCCI algebras.** It will be convenient below to work in a slightly more general setting of deformations that are not necessarily written in the form of a star product.

To define such deformations, fix an augmented (not necessarily commutative) associative algebra R, and let $R^+ \subset R$ the augmentation ideal. Given an algebra $\widehat{A}$ and an algebra imbedding $R \hookrightarrow \widehat{A}$, write $(R^+) \subset \widehat{A}$ for the two-sided ideal in $\widehat{A}$ generated by R^+. We will view R and $\widehat{A}$ as filtered algebras equipped with the R^+-adic and (R^+)-adic descending filtrations, resp., and let $\mathrm{gr}\, R$ and $\mathrm{gr}_R \widehat{A}$ denote the associated graded algebras. Thus, there is a canonical algebra map $\mathrm{gr}\, R \to \mathrm{gr}_R \widehat{A}$.

Given an algebra A and an algebra isomorphism $\phi : \widehat{A}/(R^+) \xrightarrow{\sim} A$, we say that $\widehat{A}$ is a deformation of A over R. We may view A as a graded algebra concentrated in degree zero.

Definition 7.4.1. The deformation $\widehat{A}$ of A over R is said to be a flat *free product formal deformation* if the algebra $\widehat{A}$ is complete in the (R^+)-adic topology, and the maps $\phi^{-1} : A \to \widehat{A}/(R^+)$ and $\mathrm{gr}\, R \to \mathrm{gr}_R \widehat{A}$ induce a graded algebra isomorphism

$$A * \mathrm{gr}\, R \xrightarrow{\sim} \mathrm{gr}_R \widehat{A}. \tag{7.4.2}$$

Now, fix V, a $\mathbb{Z}_+$-graded finite dimensional vector space, and let $F := TV$. Let $L \subset TV$ be a finite-dimensional vector subspace. Assuming certain favorable conditions, we can describe the equivalence classes of *all* infinite order free product deformations of an algebra of the form $A = F/(L)$ quite explicitly.

To explain this, write $\widehat{F}_t$ for the standard completion of the algebra $F_t = (TV) * \Bbbk[t]$ and $\widehat{F}_t^+ \subset \widehat{F}_t$ for the augmentation ideal. Given any linear map $\phi : L \to \widehat{F}_t^+$, we introduce a $\Bbbk[\![t]\!]$-algebra

$$A_\phi := \widehat{F}_t/(x - \phi(x))_{x \in L}. \tag{7.4.3}$$

It is clear that the projection $\widehat{F}_t \twoheadrightarrow \widehat{F}_t/\widehat{F}_t^+ = F$ induces an algebra isomorphism $A_\phi/(t) \xrightarrow{\sim} A$.

Thus, we may view the algebra A_ϕ as a 1-parameter infinite order free product deformation of A. This deformation is **not** necessarily flat, in general, i.e., the corresponding map (7.4.2) for $\widehat{A} = A_\phi$ may fail to be an isomorphism.

To formulate a sufficient condition for flatness, we recall the notion of a *noncommutative complete intersection* (NCCI), see [EG]. An algebra of the form $A = TV/(L)$ is said to be an NCCI if the two-sided ideal $J := (L)$ has the property that J/J^2 is *projective* as an $A \otimes A^{op}$-module. Moreover, such a linear subspace $L \subset TV$ is called *minimal* if $L \cap J^2 = 0$, or equivalently, L has minimal dimension (assuming J is finitely-generated).

An NCCI algebra A is known to have Hochschild dimension ≤ 2, so that $H^3(A, A\otimes A) = 0$ [EG]. Thus, free product deformations of A are unobstructed. Moreover, we have

Proposition 7.4.4. *Let $A = TV/(L)$ be an NCCI, with $L \subset TV$ minimal, and let $\phi : L \to \widehat{F}_t^+$ be a linear map. Then,*

(i) *The deformation A_ϕ defined in* (7.4.3) *is flat.*

(ii) *Any flat 1-parameter infinite order free product deformation of A is equivalent to a deformation of the form A_ϕ for an appropriate map ϕ.*

(iii) *Two deformations A_ϕ and A_ψ associated, respectively, to linear maps ϕ, $\psi \in \mathrm{Hom}_{\Bbbk}(L, \widehat{F}_t^+)$, are equivalent iff there exists a linear map $f : V \to A_t^+$, such that*

$$\pi \circ (\phi - \psi) = \Theta_f|_L. \tag{7.4.5}$$

In the last formula, we have used the notation $\pi : \widehat{F}_t \twoheadrightarrow A_t$ for the canonical quotient map and, given a linear map $f : V \to A_t^+$, write

$$\Theta_f(v_1 v_2 \cdots v_n) := \sum_{i=1}^{n} \pi(v_1 v_2 \cdots v_{i-1}) \Theta_f(v_i) \pi(v_{i+1} \cdots v_n), \quad \forall v_1, \ldots, v_n \in V.$$

Remark 7.4.6. Our proof below shows that, in the case where the image of the map ϕ is contained in the subalgebra $F_t \subset \widehat{F}_t$, we may replace the algebra A_ϕ, in (7.4.3), by $F_t/(x - \phi(x))_{x \in L}$, its non-completed counterpart. In this way, we obtain an *genuine*, rather than merely a 'formal', flat free product deformation of A.

Proof of Proposition 7.4.4. (i) By the inductive argument of [Sch, Proposition 4.2.1], one may show that the NCCI property is equivalent to the statement that the canonical surjection $\mathrm{gr}_{(L)} F \twoheadrightarrow A * L$ is an isomorphism. Thus, we conclude that the surjection $A_t \twoheadrightarrow \mathrm{gr}_{(t)} A_\phi$ is an isomorphism. It follows that A_ϕ is a flat free product deformation of A over $\Bbbk[\![t]\!]$.

(ii) For an NCCI algebra, there is a standard Anick's free resolution of A as an A-bimodule [An]:

$$0 \to A\otimes L\otimes A \to A\otimes V\otimes A \to A\otimes A \twoheadrightarrow A \to 0, \tag{7.4.7}$$

where the first map is the restriction to L of the map

$$a\otimes(v_1 v_2 \cdots v_n)\otimes b \longmapsto \sum_{i=1}^{n} a v_1 \cdot \ldots \cdot v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_n b, \quad v_i \in V,$$

and the second map has the form $a\otimes v\otimes b \mapsto av\otimes b - a\otimes vb$.

One may use Anick's resolution to compute Hochschild cohomology. We see in particular that the group $H^2(A, A\otimes A)$ is a quotient of $\mathrm{Hom}(L, A\otimes A)$. Now let A_ϕ be the deformation associated with $\phi \in \mathrm{Hom}(L, \widehat{F}_t^+)$. Then, it is easy to check that the element of $H^2(A, A\otimes A)$ corresponding to the induced first-order deformation $A_\phi/(t)^2$ is represented by the composite

$$L \to \widehat{F}_t^+ \twoheadrightarrow \widehat{F}_t^+/(\widehat{F}_t^+)^2 = F\otimes F \twoheadrightarrow A\otimes A,$$

of the map ϕ followed by two natural projections (cf. [CBEG, Lemma 10.2.1]).

Furthermore, by the inductive description of all possible star-products in §7.3, the deformations A_ϕ must exhaust all possible deformations (note that all possible classes of $H^2(A, A^{\otimes m})$ at every step of the way are attained, which is as it must be, since $H^3(A, A\otimes A) = 0$).

(iii) At the first-order stage, we see from Anick's resolution (7.4.7) that two elements of the space $\mathrm{Hom}_k(L, A\otimes A)$ yield the same cohomology class in $H^2(A, A\otimes A)$ iff they differ by Θ_f from condition (7.4.5) modulo $(t)^2$, for some f. Also, since $H^1(A, A\otimes A) = 0$, it follows as in the classical setting that there are no nontrivial gauge-equivalence relations. Hence, by the inductive construction of all free-product deformations from §7.3, we deduce the desired result. $\square$

Remark 7.4.8. (i) In general, given an arbitrary algebra A such that $H^3(A, A\otimes A) = 0$, one can show that there exists a 'versal' free product deformations of A. The base of such a versal deformation is a completed tensor algebra of the vector space H^*, where $H := H^2(A, A\otimes A)$.

(ii) Proposition 7.4.4 may be generalized easily to the case where the ground field $\Bbbk$ is replaced by a ground ring R, a finite dimensional semisimple $\Bbbk$-algebra, as in [EG]. Such a generalized version of Proposition 7.4.4 applies to preprojective algebras of non Dynkin quivers, in particular. Thus the proposition may be viewed as a generalization of [CBEG], Theorem 10.1.3.

(iii) Let $\pi_1(X)$ be the fundamental group of a compact oriented Riemann surface X of genus ≥ 1. The group algebra $\Bbbk[\pi_1(X)]$ may be thought of as a multiplicative analogue of the preprojective algebra of a non-Dynkin quiver (being non-Dynkin corresponds to the condition that the Euler characteristic of X be nonpositive). Accordingly, there is a similar construction of free product deformations of the group algebra as follows.

Let g be the genus of X, and write $a_1, \ldots, a_g, b_1, \ldots, b_g$ for the standard loops around the handles, which generate $\pi_1(X)$. The group $\pi_1(X)$ is the quotient of Γ, the free group generated by the letters a_i, b_i, by the normal subgroup generated by the following element:

$$\gamma := (a_1 b_1 a_1^{-1} b_1^{-1}) \cdot \ldots \cdot (a_g b_g a_g^{-1} b_g^{-1}) \in \Gamma.$$

To construct free product deformations of the group algebra $\Bbbk[\pi_1(X)]$, we put $F := \Bbbk[\Gamma]$. The algebra F is a 'multiplicative analogue' of a free algebra. To any element $u \in 1 + F_t^+$, we associate an algebra $A_u := F_t/(\gamma - u)$.

There is a 'multiplicative analogue' of Proposition 7.4.4, saying that the algebra A_u gives a flat free-product deformation of the group algebra $\Bbbk[\pi_1(X)]$, and moreover that these are all such deformations up to equivalence. One may prove this result by using the fact that the prounipotent completion of $\Bbbk[\pi_1(X)]$ is isomorphic to a completion of an algebra of the form $\Bbbk\langle x_1, \ldots, x_g, y_1, \ldots, y_g\rangle/([x_1, y_1] + \ldots + [x_g, y_g])$.

This example may be generalized to the situation of orbifold surfaces of nonpositive Euler characteristic (the latter are also NCCI algebras).

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